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Global stability of two-dimensional and axisymmetric Euler flows

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This paper is concerned with the stability of steady inviscid flows with closed streamlines. In increasing order of complexity we look at two-dimensional planar flows, poloidal (r, z) flows, and swirling recirculating flows. In each case we examine the relationship between Arnol'd's variational approach to stability, Moffatt's magnetic relaxation technique, and a more recent relaxation procedure developed by Vallis *et al.* We start with two-dimensional (x, y) flows. Here we show that Moffatt's relaxation procedure will, under a wide range of circumstances, produce Euler flows which are stable. The physical reasons for this are discussed in the context of the well-known membrane analogy. We also show that there is a close relationship between Hamilton's principle and magnetic relaxation. Next, we examine poloidal flows. Here we find that, by and large, our planar results also hold true for axisymmetric flows. In particular, magnetic relaxation once again provides stable Euler flows. Finally, we consider swirling recirculating flows. It transpires that the introduction of swirl has a profound effect on stability. In particular, the flows produced by magnetic relaxation are no longer stable. Indeed, we show that all swirling recirculating Euler flows are potentially unstable to the extent that they fail to satisfy Arnol'd's stability criterion. This is, perhaps, not surprising, as all swirling recirculating flows include regions where the angular momentum decreases with radius and we would intuitively expect such flows to be prone to a centrifugal instability. The paper concludes with a discussion of marginally unstable modes in swirling flows. In particular, we examine the extent to which Rayleigh's original ideas on stability may be generalized, through the use of the Routhian, to include flows with a non-zero recirculation.

1. Introduction

We are interested here in the stability of steady incompressible inviscid flows (Euler flows). In particular, we make use of the variational techniques of Arnol'd (1966*a*) and of Moffatt (1985, 1986, 1990). Arnol'd's well-known stability theorem provides a sufficient criterion for an inviscid flow to be stable. It establishes stability by advecting the vortex lines of the Euler flow by a virtual displacement field, $\boldsymbol{\eta}(\boldsymbol{x})$, which satisfies $\nabla \cdot \boldsymbol{\eta} = 0$, and $\boldsymbol{\eta} \cdot \mathrm{d}\boldsymbol{s} = 0$ on the boundary. Arnol'd showed that, when the velocity field $\boldsymbol{u}$ is a steady state, the kinetic energy, E, of the flow has a stationary value, $\delta^1 E = 0$. Moreover, if this stationary value is either a maximum or a minimum, then the flow is linearly stable. That is, a sufficient condition for stability is that $\delta^2 E$ be of definite sign for all possible displacement fields $\boldsymbol{\eta}$. Note that failure to satisfy Arnol'd's criterion does not imply instability, but merely 'potential instability'.

Arnol'd was also aware of magnetic relaxation as a vehicle for providing information about Euler flows. However, this technique has been developed most notably by Moffatt (1985, 1986, 1990). Consider a viscous perfectly conducting fluid which initially contains a magnetic field $\boldsymbol{B}(\boldsymbol{x})$ or arbitrary structure. (We merely require that $\boldsymbol{B}\cdot \mathrm{d}\boldsymbol{s} = 0$ on the boundary.) In general the Lorentz force $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ will be rotational and so induce motion. It is not difficult to show that, as long as $\boldsymbol{u} \neq \boldsymbol{0}$, the total energy of the system decreases monotonically due to viscous dissipation. However, since the fluid is perfectly conducting, $\boldsymbol{B}$ is smoothly advected by the velocity field, and so retains its topological structure. In many cases this structure is sufficient to ensure that the magnetic energy cannot tend to zero (Moffatt 1985), and so the end result of this 'relaxation' process is a magnetostatic equilibrium, with $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$ and the final $\boldsymbol{B}$-field retaining the same topological structure as the initial $\boldsymbol{B}$-field. We now note that there is a well-known analogy between the magnetostatic equilibrium

$$0 = \boldsymbol{j} \times \boldsymbol{B} - \nabla p, \quad \boldsymbol{j} = \nabla \times \boldsymbol{B} \tag{1.1}$$

and the Euler equations for steady flow

$$0 = \boldsymbol{u} \times \boldsymbol{\omega} - \nabla H, \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}. \tag{1.2}$$

Here H is Bernoulli's constant $(p/\rho + \frac{1}{2}\boldsymbol{u}^2)$, $\boldsymbol{j}$ is the current density and p is the pressure. (We have taken the permeability, μ_0, to be equal to unity for convenience.) The analogy is between the variables

$$\boldsymbol{B} \leftrightarrow \boldsymbol{u}, \quad \boldsymbol{j} \leftrightarrow \boldsymbol{\omega} \quad \text{and} \quad p \leftrightarrow -H.$$

Thus, magnetic relaxation provides an Euler flow whose streamlines have the same topology as the initial magnetic field. This not only allows one to infer the existence of Euler flows under a variety of conditions, but also provides an algorithm for computing such flows.

Unfortunately, as pointed out by Moffatt (1985), the Euler flows produced by this procedure are not, in general, stable. The reason is that perturbations about the steady state are governed, in the case of an Euler flow, by advection of the $\boldsymbol{\omega}$-lines, whereas the magnetostatic equilibrium was approached by advection of the $\boldsymbol{B}$-lines (which correspond to $\boldsymbol{u}$-lines in the analogy). Indeed, Moffatt (1990) gives simple examples of unstable Euler flows produced by this procedure, and in particular shows that a common feature of magnetic relaxation is the formation of current sheets, corresponding to unstable vortex sheets in the $\boldsymbol{B} \leftrightarrow \boldsymbol{u}$ analogy.

This difficulty has been resolved, to some degree, by Vallis, Carnevale & Young (1989) who developed an elegant relaxation procedure which combines elements of both Arnol'd's stability criteria and magnetic relaxation. They introduced a form of 'modified dynamics', described by

$$\partial \boldsymbol{\omega}/\partial t = \nabla \times (\boldsymbol{u}^* \times \boldsymbol{\omega}),$$
$$\boldsymbol{u}^* = \boldsymbol{u} + \lambda\, \partial \boldsymbol{u}/\partial t,$$

where λ is a constant or a function of time. Here we may think of $\boldsymbol{u}^*$ as a continuously evolving form of Arnol'd's displacement field which is functionally related to the instantaneous velocity, $\boldsymbol{u}$. These equations have a number of interesting properties. First, the vorticity is smoothly advected by $\boldsymbol{u}^*$. It follows that all the normal invariants of the velocity field, such as circulation and helicity, are conserved. Secondly, the energy of the flow monotonically decreases or increases according to,

$$\dot{E}(t) = \frac{\partial}{\partial t}\int_V \tfrac{1}{2}\boldsymbol{u}^2\, \mathrm{d}V = -\lambda \int_V \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^2 \mathrm{d}V.$$

Thirdly, the change in energy ceases only when the flow reaches a steady state, and in this case the modified dynamics revert to the conventional Euler equations. It is always possible that the modified dynamics will produce the trivial results $E = 0$ or $E \rightarrow \infty$.

However, if it does converge to a solution with finite energy, then in general this will be either a local minimum or maximum in energy and so the resulting flow will be stable by Arnol'd's criterion. This technique is particularly powerful for two-dimensional flows since E is bounded from above when the vorticity is conserved (see Vallis *et al.* 1989) and so the scheme is guaranteed to produce a non-trivial result when $\lambda < 0$. However, in three dimensions, one cannot, in general, place bounds on the kinetic energy. (One important exception will be described later.) Moreover, some researchers have suggested that the Euler equations can produce finite-time singularities and, if this is so, it would be surprising if the modified dynamics did not. We may regard, therefore, the relaxation procedures of Moffatt and Vallis *et al.* as complementary, each with its own advantages and limitations.

In this paper we look at the relationship between Moffatt's relaxation procedure, Arnol'd's stability theorem, and the modified dynamics of Vallis *et al.* In particular, we use the results of Mestel (1989) to bridge the gap between these concepts. The discussion is restricted to two-dimensional and axisymmetric flows, including planar flows, axisymmetric flows without swirl, and finally axisymmetric flows with swirl. In all cases, the motion is restricted to a simply connected domain V bounded by a surface S.

The plan of the paper is as follows. We start with two-dimensional flows in the (x, y)-plane. Here we consider steady flows in which the vorticity is smoothly distributed throughout V and is non-zero at every point. This excludes, for example, isolated vortex patches and flows containing vortex sheets. In §3 we show that, rather surprisingly, magnetic relaxation does indeed produce Euler flows which are stable to two-dimensional disturbances. The physical reasons for this are discussed in §4 using the well-known membrane analogy. We also show, in §5, that the extremal nature of the kinetic energy under a perturbation of the streamlines follows directly from Hamilton's principle.

Next, we extend the analysis to axisymmetric flows described using a cylindrical polar coordinate system (r, θ, z). We start, in §7, with poloidal flows (axisymmetric flows without swirl). These exhibit essentially the same characteristics as planar flows. In particular, the stability results of §§3–5 carry over with little modification. As before, magnetic relaxation generally provides a stable Euler flow, subject to certain restrictions.

Finally, we consider flows where both the swirl, u_θ, and the poloidal recirculation $(u_r, 0, u_z)$ are non-zero. It transpires that the introduction of swirl has a profound effect on the stability analysis. In fact, all such flows are potentially unstable, to the extent that they fail to meet Arnol'd's criterion. We illustrate this with two particular examples. The fact that swirling recirculating flows are potentially unstable is, perhaps, not surprising. They inevitably contain regions where the angular momentum decreases with radius, and intuitively one would expect such flows to be prone to a Rayleigh-like centrifugal instability. The extent to which Rayleigh's original ideas on stability can be generalized to encompass recirculating flows is also discussed.

It is convenient to introduced some elementary concepts and notation at this point. In two-dimensional flows we define the streamfunction via $\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \hat{\boldsymbol{e}}_z)$. The steady Euler equation (1.2) then reduces to the statement $\omega = -H'(\psi)$, and so the steady state is governed by the equation

$$\nabla^2 \psi = -\omega(\psi) = H'(\psi). \tag{1.3}$$

When the flow is unsteady, ω is no longer constant along each streamline, but in two-dimensional flows it is an advected quantity, in the sense that

$$\mathrm{D}\omega/\mathrm{D}t = 0.$$

It follows that we have a class of integral invariants of the form

$$I = \int_{V_\omega} f(\omega)\,\mathrm{d}V, \tag{1.4}$$

where f is an arbitrary function of ω, and V_ω is any material volume enclosed by the surface $\omega =$ constant. The simplest invariant is V_ω itself, which has been termed the signature function (see Moffatt 1990). This plays a central role in any isovortical relaxation process, such as that of Vallis *et al.*, since conservation of V_ω restricts the potential outcome of the relaxation. Note that, in two dimensions, conservation of vorticity places an upper bound on the kinetic energy of the flow. In particular, if λ_0 is the lowest eigenvalue of

$$\nabla^2\phi + \lambda\phi = 0, \quad \phi = 0 \quad \text{on} \quad S$$

then we have the Poincaré inequality (see Vallis *et al.* 1989)

$$E = \int_V \tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V \leqslant \frac{1}{2\lambda_0}\int_V \omega^2\,\mathrm{d}V. \tag{1.5}$$

This ensures non-trivial results from the modified dynamics of Vallis *et al.* in two dimensions.

Finally, we shall find it convenient to introduce the idea of recirculation time for steady two-dimensional flows. Let $\tau(\psi)$ be the time it takes a fluid particle to pass once around the streamline C_ψ. That is, if l is the distance measured along C_ψ from some datum,

$$\tau(\psi) = \oint_{C_\psi} \mathrm{d}l/|\boldsymbol{u}|. \tag{1.6}$$

In fact, $\tau(\psi)$ is simply related to the volume enclosed by C_ψ. To see why this is so, consider a streamtube bounded by the two streamlines ψ and $\psi+\delta\psi$. Let δn be the separation of the two streamlines. Then the volume enclosed by the streamtube is

$$V_\psi(\psi+\delta\psi) - V_\psi(\psi) = -\oint_{C_\psi} \delta n\,\mathrm{d}l,$$

where V_ψ is the volume enclosed by C_ψ. If we now recall that $|\boldsymbol{u}| = |\boldsymbol{\nabla}\psi| = \delta\psi/\delta n$, this reduces to

$$\tau(\psi) = -\mathrm{d}V_\psi/\mathrm{d}\psi. \tag{1.7}$$

We shall find this relationship particularly useful when applying Hamilton's principle to magnetic relaxation. First, however, we shall review Arnol'd's stability theorem.

2. A review of Arnol'd's stability criterion

As a prelude to investigating the stability of two-dimensional flows it is convenient to review Arnol'd's stability theorem. We shall give here only the briefest of accounts. More details may be found in Arnol'd (1966*a*), Moffatt (1986) and Mestel (1989).

Our starting point is to introduce the idea of a virtual displacement of an Euler flow. This is discussed in detail in Moffatt (1986). However, it is worth reviewing this concept here. Virtual displacements are familiar from conventional Lagrangian dynamics as perturbations of the generalized coordinates of a mechanical system. Such perturbations are quite arbitrary, except to the extent that they must satisfy all the system constraints. Of course, the application of a virtual displacement does not imply that, through the action of some perturbing force, the system actually moves to a new configuration. Such displacements are merely a device for exploring (potential) instantaneous system configurations 'adjacent' to the true system configuration.

In the context of a steady Euler flow, $\boldsymbol{u}(\boldsymbol{x})$, we need to introduce a continuous virtual displacement field. The constraint which this field must observe is conservation of volume (Lanczos 1970). This may be enforced by supposing that the displacement occurs through the action of a solenoidal velocity field, $\boldsymbol{v}(\boldsymbol{x})$, applied for a short time τ. Let $\zeta(\boldsymbol{x})$ be the displacement of a fluid particle initially at $\boldsymbol{x}$, and $\boldsymbol{\eta}(\boldsymbol{x})$ be defined by $\boldsymbol{\eta} = \boldsymbol{v}\tau$. To first order, ζ and $\boldsymbol{\eta}$ are equal. At second order curvature of the $\boldsymbol{v}$-lines is important, and (see Moffatt 1986)

$$\zeta = \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta} + O(\tau^3).$$

In line with previous investigators, we shall refer to $\boldsymbol{\eta}$ as a 'kinetically admissible' virtual displacement field (although $\boldsymbol{\eta}$ is strictly only the displacement to first order). By definition,

$$\nabla\cdot\boldsymbol{\eta} = 0, \quad \boldsymbol{\eta}\cdot\mathrm{d}\boldsymbol{S} = 0 \quad \text{on} \quad S.$$

Now in Arnol'd's theorem we explore system configurations (i.e. velocity fields) adjacent to the steady Euler flow $\boldsymbol{u}(\boldsymbol{x})$. However, these perturbed configurations are of a particular type. They must all have the same vortex-line topology as the initial Euler flow. In the terminology of Vallis *et al.* (1989), our perturbations are restricted to an *isovortical sheet* in phase space. The rationale for this is as follows. The Euler equations describe flow in an infinite-dimensional phase space. This space consists of subdomains (isovortical sheets) on which the vorticity configurations can be mapped one to another by a smooth displacement of the vortex lines. An unsteady Euler flow is constrained to follow a constant-energy contour on just such an isovortical sheet. Now steady Euler flows represent stationary points on these sheets, at which $\delta^1 E = 0$. If this point is also an extremum in energy, so that the constant-energy contours are locally elliptic, then it seems plausible that the flow is stable, in that an isovortically perturbed flow will subsequently evolve on a constant-energy contour which always lies close to the stationary point. However, if the equilibrium represents a saddle point on the sheet, then adjacent energy contours can diverge, so that an isovortically perturbed flow is no longer constrained to stay close to the initial equilibrium position. Arnol'd therefore associated stability with extremums in E under an isovortical perturbation.

An isovortical perturbation can be achieved by applying our virtual displacement field, $\boldsymbol{\eta}$, to the vortex lines of the steady Euler flow. That is, we consider the vortex lines to be advected by an arbitrary solenoidal velocity field $\boldsymbol{v}$ for a short time τ. Note however, that we are not implying that the flow actually moves from one state to the other, say under the action of some force. Note also that any of these adjacent (perturbed) velocity fields could be taken as an initial condition for a new unsteady flow. To that extent, we may consider that we have created a new pressure field during the virtual displacement. This new pressure is determined by the divergence of the unsteady Euler equation. However, we shall have no particular need to calculate this perturbed pressure.

In subsequent sections we shall find it useful to perturb Euler flows in other ways. For example, rather than explore adjacent system configurations on the same isovortical sheet, we shall consider adjacent solenoidal velocity fields obtained by displacing the $\boldsymbol{u}$-lines. Although not relevant to Arnol'd's theorem, this second type of perturbation is useful in discussing magnetic relaxation. To keep the arguments general, therefore, we start by applying $\boldsymbol{\eta}$ to an arbitrary scalar or vector field. In practice, this field may be $\boldsymbol{u}, \psi, \omega$ or, as we shall see, angular momentum. Scalar fields $f(\boldsymbol{x})$ which are advected by $\boldsymbol{v}$ are perturbed according to

$$\partial f/\partial t = -\boldsymbol{v}\cdot\nabla f, \quad 0 \leqslant t \leqslant \tau,$$

from which the first-order perturbation in f is

$$\delta^1 f = -\boldsymbol{\eta}\cdot\nabla f. \tag{2.1}$$

Substituting this expression back into the advection equation and reintegrating gives the second-order perturbation

$$\delta^2 f = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\delta^1 f). \tag{2.2}$$

We shall return to these expressions later. Vector fields $\boldsymbol{A}(\boldsymbol{x})$ which are advected by $\boldsymbol{v}$ obey the 'frozen-field' equation,

$$\partial \boldsymbol{A}/\partial t = \nabla\times(\boldsymbol{v}\times\boldsymbol{A})$$

and so the first- and second-order perturbations in $\boldsymbol{A}$ are

$$\delta^1\boldsymbol{A} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{A}), \tag{2.3}$$

$$\delta^2\boldsymbol{A} = \tfrac{1}{2}\nabla\times(\boldsymbol{\eta}\times\delta^1\boldsymbol{A}). \tag{2.4}$$

We shall talk about vortex lines being perturbed, in which case $\boldsymbol{A} = \boldsymbol{\omega}$, or else streamlines being perturbed, in which case $\boldsymbol{A} = \boldsymbol{u}$. (We shall delay our discussion of perturbing the $\boldsymbol{u}$-lines to the section on magnetic relaxation.) In Arnol'd's theory we apply a virtual displacement field to the vortex lines and examine the consequent change in kinetic energy. Thus,

$$\delta^1\boldsymbol{\omega} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}), \quad \delta^1\boldsymbol{u} = \boldsymbol{\eta}\times\boldsymbol{\omega}+\nabla\phi_1,$$

$$\delta^2\boldsymbol{\omega} = \tfrac{1}{2}\nabla\times(\boldsymbol{\eta}\times\delta^1\boldsymbol{\omega}), \quad \delta^2\boldsymbol{u} = \tfrac{1}{2}\boldsymbol{\eta}\times\delta^1\boldsymbol{\omega}+\nabla\phi_2,$$

where ϕ_1 and ϕ_2 are chosen to ensure $\delta^1\boldsymbol{u}$ and $\delta^2\boldsymbol{u}$ are solenoidal. The first-order change in E is

$$\delta^1 E = \int_V \boldsymbol{u}\cdot\delta^1\boldsymbol{u}\,\mathrm{d}V$$

and it is readily confirmed that $\delta^1 E$ vanishes when $\boldsymbol{u}$ is a steady velocity field. The second-order perturbation in E is

$$\delta^2 E = \int_V [\tfrac{1}{2}(\delta^1\boldsymbol{u})^2 + \boldsymbol{u}\cdot\delta^2\boldsymbol{u}]\,\mathrm{d}V,$$

which may be rewritten in the more useful form

$$\delta^2 E = \frac{1}{2}\int_V [(\delta^1\boldsymbol{u})^2 + \boldsymbol{\eta}\cdot(\delta^1\boldsymbol{\omega}\times\boldsymbol{u})]\,\mathrm{d}V. \tag{2.5}$$

Arnol'd's theory then says that the Euler flow is stable if $\delta^2 E$ is positive definite or negative definite; that is, E is a maximum or a minimum for the Euler flow. Flows which fail to satisfy Arnol'd's criterion are not necessarily unstable, so following Mestel (1989) we describe these as merely 'potentially unstable'. In fact, we shall give an example of a stable flow which fails to satisfy Arnol'd's criterion in the next section.

We now restrict ourselves to flows *and* perturbations which are two-dimensional. The first-order perturbation in vorticity is then given by (2.1),

$$\delta^1\omega = -\boldsymbol{\eta}\cdot\nabla\omega = -\omega'(\psi)\,\boldsymbol{\eta}\cdot\nabla\psi \tag{2.6}$$

so (2.5) simplifies to

$$\delta^2 E = \frac{1}{2}\int_V [(\delta^1 \boldsymbol{u})^2 - \omega'(\psi)(\boldsymbol{\eta}\cdot\nabla\psi)^2]\,\mathrm{d}V.$$

Next, we introduce an important restriction. We shall assume that $\omega'(\psi)$ is non-zero everywhere in V. This implies that we are excluding vortex patches where ω is zero over finite regions of the flow field. In this case we have

$$\delta^2 E = \frac{1}{2}\int_V [(\nabla\phi)^2 - (\nabla^2\phi)^2/\omega'(\psi)]\,\mathrm{d}V, \tag{2.7}$$

where ϕ is the first-order perturbation of the streamfunction, $\delta^1\psi$. Expression (2.7) is the most useful form for $\delta^2 E$ in two-dimensional problems. Clearly, when $\omega'(\psi) < 0$, the corresponding Euler flow is stable and represents a minimum in E. When $\omega'(\psi) > 0$ the two terms in (2.7) are of opposite sign with the second term dominant for perturbations with a short lengthscale. Consequently, to ensure stability of flows with $\omega'(\psi) > 0$ we must demonstrate that $\delta^2 E < 0$ for all ϕ. The Euler flow then represents a maximum in energy. Given $\omega(\psi)$, we may locate the maximum value of $\delta^2 E$ using the calculus of variations (see, for example, Williams 1980). The simplest approach is to normalize $\delta^2 E$ by dividing by $\frac{1}{2}\int (\nabla\phi)^2\,\mathrm{d}V$ and look for the minimum of the ratio. The resulting eigenvalue problem then gives (see Mestel 1989)

$$\delta^2 E_{max} = (1-\lambda_0)\frac{1}{2}\int_V (\nabla\phi)^2\,\mathrm{d}V, \tag{2.8}$$

where λ_0 is the minimum eigenvalue of

$$\nabla^2\phi + \lambda\omega'(\psi)\phi = 0, \quad \phi = 0 \quad \text{on} \quad S. \tag{2.9}$$

Consequently, as Mestel (1989) noted, the two-dimensional stability of two-dimensional flows is guaranteed if either $\omega'(\psi) < 0$, or else $\omega'(\psi) > 0$ and the minimum eigenvalue of (2.9) is greater than unity. If $\omega'(\psi)$ changes sign in V then $\delta^2 E$ may be made positive or negative by choosing ϕ to be highly localized and applied in a region where $\omega'(\psi)$ has the appropriate sign. Consequently, flows in which $\omega'(\psi)$ changes sign invariably fail to meet Arnol'd's stability criterion.

We conclude this review of Arnol'd's analysis by noting that (2.7) may be derived by a more direct route if we restrict ourselves to two-dimensional flows from the outset. Following Arnol'd (1966*b*) we introduce the functional

$$A(\Psi) = \int_V \left[\tfrac{1}{2}(\nabla\Psi)^2 - \int_0^{\Omega} \psi_0(\omega)\,\mathrm{d}\omega\right]\mathrm{d}V, \tag{2.10}$$

where Ψ is the streamfunction of an unsteady flow, Ω is the vorticity, $\Omega = -\nabla^2\Psi$, and $\psi_0(\omega)$ is the inverse function of $\omega(\psi)$, the vorticity distribution of some steady Euler flow. (Note that, in order to define $\psi_0(\omega)$, we require $\omega'(\psi) \neq 0$ which is consistent with our previous restriction on ω.) By virtue of (1.4), and of conservation of energy, the functional A is a conserved quantity. Now suppose Ψ is close to ψ, $\Psi = \psi + \phi$. Then we may show that A is stationary when $\Psi = \psi$ and that the second-order perturbation in A is

$$\delta^2 A = \frac{1}{2}\int_V [(\nabla\phi)^2 - (\nabla^2\phi)^2/\omega'(\psi)]\,\mathrm{d}V, \tag{2.11}$$

which is the same as our expression for $\delta^2 E$. However, A is conserved by an inviscid flow and so, in the linear approximation, $\delta^2 A$ is also conserved. If the flow is to be unstable, then $\int (\nabla\phi)^2\,\mathrm{d}V$ must become large while $\delta^2 A$ remains constant. Conversely, provided we can bound $\delta^2 A$ away from zero, for a constant $\int (\nabla\phi)^2\,\mathrm{d}V$, the flow must remain stable. Consequently, a sufficient condition for formal stability is that $\delta^2 A$ is positive definite or negative definite, which is precisely the same as the previous criterion.

3. Two-dimensional Euler flows produced by magnetic relaxation

We now compare Arnol'd's stability criterion with the process of magnetic relaxation. It is well known that the magnetic energy of a magnetostatic equilibrium is stationary with respect to perturbations of the $\boldsymbol{B}$-lines (see, for example, Moffatt 1990). In addition, the equilibrium is stable if the energy is minimal. The physical nature of magnetic relaxation suggests that, in general, it will produce $\boldsymbol{B}$-fields in which the energy is indeed a minimum, and so these solutions must be stable. Now consider the corresponding Euler flows. We expect that, from the exact analogy between $\boldsymbol{B}$ and $\boldsymbol{u}$, flows produced by magnetic relaxation will have a kinetic energy which is stationary (and indeed minimal) with respect to a perturbation of the $\boldsymbol{u}$-lines.

Now perturbating the $\boldsymbol{u}$-lines of an Euler flow may seem somewhat unphysical, since it is the vortex lines which are advected in the unsteady Euler equations. Some careful explanation is therefore required at this point. Let $\boldsymbol{B}_0$ be a magnetostatic equilibrium formed by magnetic relaxation, and $\boldsymbol{u}_0$ be the equivalent Euler flow. The analogy between $\boldsymbol{u}$ and $\boldsymbol{B}$ holds only at equilibrium, and not for the perturbed states adjacent to $\boldsymbol{B}_0$ and $\boldsymbol{u}_0$. However, magnetic relaxation tells us that if, by advecting the $\boldsymbol{B}$-lines, we explore all the topologically equivalent solenoidal $\boldsymbol{B}$-fields adjacent to $\boldsymbol{B}_0$, then we should find that the magnetic energy is minimal at equilibrium. Now, by perturbing the $\boldsymbol{u}$-lines of the Euler flow, $\boldsymbol{u}_0$, we are merely finding the solenoidal velocity fields identical to the perturbed $\boldsymbol{B}$-fields. It follows that the kinetic energy of the Euler flow will be minimal under this (unphysical) perturbation.

Clearly, no particular physical significance should be attached to the process of perturbing the $\boldsymbol{u}$-lines. In particular, the perturbed velocity fields need not be 'dynamically accessible' from the equilibrium velocity field. Curiously, however, it turns out that this perturbation does have a meaningful interpretation in the framework of Lagrangian dynamics, in terms of perturbing individual particle trajectories. This is discussed in §6.

Let us now focus on two-dimensional Euler flows. Applying a virtual displacement to the $\boldsymbol{u}$-lines in two dimensions is equivalent to advecting ψ, the streamfunction. From (2.1) and (2.2) we have,

$$\mathrm{d}^1\psi = -\boldsymbol{\eta}\cdot\nabla\psi, \tag{3.1}$$

$$\mathrm{d}^2\psi = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\mathrm{d}^1\psi), \tag{3.2}$$

where $\boldsymbol{\eta}$ is our virtual displacement field, ψ is the streamfunction of the steady Euler flow, and we use the symbol d rather than δ to distinguish between advection of the $\boldsymbol{u}$-lines and advection of the $\boldsymbol{\omega}$-lines. The first-order perturbation in E is then

$$\mathrm{d}^1 E = \int_V \nabla\psi\cdot\nabla(\mathrm{d}^1\psi)\,\mathrm{d}V.$$

However, if we note that

$$\nabla\psi\cdot\nabla(\mathrm{d}^1\psi) = \nabla\cdot[\mathrm{d}^1\psi\nabla\psi+\boldsymbol{\eta}H]$$

it is clear that $d^1E = 0$, as expected. The second-order perturbation in E is

$$d^2E = \frac{1}{2}\int_V [(\nabla(d^1\psi))^2 + 2\nabla\psi\cdot\nabla(d^2\psi)]\,dV.$$

If we now substitute for $d^2\psi$ in terms of $d^1\psi$, and invoke the divergence theorem, we obtain

$$d^2E = \frac{1}{2}\int_V [(\nabla\phi)^2 - \omega(\boldsymbol{\eta}\cdot\nabla\phi)]\,dV,$$

where, for brevity, we write $\phi = d^1\psi$. However, since ω is a function of ψ alone, we have

$$\omega(\boldsymbol{\eta}\cdot\nabla\phi) = \nabla\cdot(\phi\omega\boldsymbol{\eta}) + \phi^2\omega'(\psi)$$

from which we obtain a simpler expression for d^2E:

$$d^2E = \frac{1}{2}\int_V [(\nabla\phi)^2 - \omega'(\psi)\,\phi^2]\,dV. \tag{3.3}$$

At this point we might note that we have implicitly assumed that $\omega'(\psi)$ is finite everywhere, and so we are excluding flows which contain vortex sheets. Now we know that magnetic relaxation will, in general, give Euler flows in which d^2E is positive definite. Let us consider which vorticity distributions are compatible with this. Clearly, Euler flows with $\omega'(\psi) < 0$ guarantee $d^2E > 0$. If $\omega'(\psi) > 0$ then the situation is more complicated. Here we require d^2E to be bounded from below by a positive number. This again leads us to a variational problem. Consider the functional

$$I(\phi) = \frac{\frac{1}{2}\int [(\nabla\phi)^2 - f\phi^2]\,dV}{\frac{1}{2}\int [w\phi^2]\,dV}, \quad \phi = 0 \quad \text{on} \quad S, \tag{3.4a}$$

where f and w are arbitrary functions (except that $w > 0$). We know (see, for example, Williams 1980) that I has a minimum value when ϕ is the eigenfunction corresponding to the least eigenvalue, λ, of

$$\nabla^2\phi + f\phi + \lambda w\phi = 0, \quad \phi = 0 \quad \text{on} \quad S. \tag{3.4b}$$

The minimum value of I is λ_0, the smallest eigenvalue of (3.4b). Now suppose that we normalize d^2E by dividing it by $\frac{1}{2}\int \omega'(\psi)\,\phi^2\,dV$ (for $\omega'(\psi) > 0$). It follows from (3.4a) and (3.4b) that d^2E is positive definite if the least eigenvalue of

$$\nabla^2\phi + (1+\lambda)\,\omega'(\psi)\,\phi = 0, \quad \phi = 0 \quad \text{on} \quad S \tag{3.5}$$

is greater than zero. However, this is precisely the same eigenvalue problem as (2.9), which is the condition required to ensure δ^2E is always negative (for $\omega'(\psi) > 0$). In summary then, if $\omega'(\psi) < 0$, both δ^2E and d^2E are positive definite. When $\omega'(\psi) > 0$, however, then either both δ^2E and d^2E are indefinite in sign, or else δ^2E is negative definite and d^2E is positive definite. It follows that the two-dimensional Euler flows produced by magnetic relaxation, which in general satisfy $d^2E > 0$, also satisfy Arnol'd's criterion, and so are stable to two-dimensional perturbations.

At this point it is, perhaps, worth summarizing the restrictions we have placed on $\omega(\psi)$. For the above statements to hold true we require:

(a) $\omega'(\psi) \neq 0$ (no vortex patches);

(b) $\omega'(\psi)$ finite (no vortex sheets);
(c) $\omega'(\psi)$ does not change sign.

When these conditions are satisfied, we may take the positive definiteness of d^2E as a sufficient condition for stability. Mestel (1989) has previously noted that the integral in (3.3) provided an alternative formulation of Arnol'd's criterion, and that its simple familiar form lends itself to developing more detailed stability theorems. Mestel was not concerned with magnetic relaxation, however, but rather identified

$$J = \int_V [(\nabla\phi)^2 - \omega'(\psi)\,\phi^2]\,\mathrm{d}V$$

as the second variation of the functional

$$M(\Psi) = \int_V \left[\frac{1}{2}(\nabla\Psi)^2 - \int_0^{\Psi} \omega(\psi)\,\mathrm{d}\psi\right]\mathrm{d}V, \tag{3.6}$$

which is stationary when $\Psi = \psi$, the streamfunction of an Euler flow. This functional bears an interesting resemblance to Arnol'd's functional (2.10), but is not a conserved quantity, unlike $A(\Psi)$.

Now our claim that Euler flows produced by magnetic relaxation are stable is in marked contrast to Moffatt's (1990) counter examples. Moffatt gave several instances of unstable Euler flows produced by relaxation. Some of Moffatt's examples are axisymmetric and contain swirl, and we shall return to these flows later. However, the two-dimensional examples given by Moffatt do appear to contradict our main conclusion. But these flows involved the formation of vortex sheets from saddle points in the initial $\boldsymbol{B}$-field, and we have already noted that our analysis is not valid in such cases. There is, therefore, no conflict between our analysis and that of Moffatt.

If we choose to normalize d^2E in a different way we may derive an alternative eigenvalue problem to (3.5). Dividing d^2E by $\frac{1}{2}\int \phi^2\,\mathrm{d}V$, and applying (3.4*a*) and (3.4*b*), tells us that d^2E is positive definite provided

$$\nabla^2\phi + \omega'(\psi)\,\phi + \lambda\phi = 0, \quad \phi = 0 \quad \text{on} \quad S \tag{3.7}$$

produces positive eigenvalues. This formulation has two advantages over (3.5). First, it provides a single test for the sign of d^2E, both for $\omega'(\psi) > 0$ and $\omega'(\psi) < 0$. Secondly, it has a familiar form which appears in a variety of other physical problems, as we shall see later.

Let us now illustrate some of these observations using a very simple example. Suppose the boundary, S, is the circle $r = R$. Then (1.3) demands that the flow be of the form $\boldsymbol{u} = u_\theta(r)\,\hat{\boldsymbol{e}}_\theta$ (using polar coordinates). Suppose we narrow the choice of $\boldsymbol{u}$ and specify $\omega'(\psi) = \alpha^2$. Then, in terms of Bessel functions, (1.3) gives

$$\psi = C[J_0(\alpha r) - J_0(\alpha R)], \quad u_\theta = \alpha C J_1(\alpha r), \quad \omega = \alpha^2 C J_0(\alpha r),$$

where C is an arbitrary amplitude. From (2.7), advection of the vortex lines perturbs E according to

$$\delta^2 E = \frac{1}{2}\int_V [(\nabla\phi)^2 - (\nabla^2\phi)^2/\alpha^2]\,\mathrm{d}V.$$

This is bounded above by

$$\delta^2 E_{max} = (1-\lambda_0^*)\frac{1}{2}\int_V (\nabla\phi)^2\,\mathrm{d}V,$$

where λ_0^* is the minimum eigenvalue of (2.9),

$$\nabla^2\phi + \lambda^* \alpha^2 \phi = 0, \quad \phi = 0 \quad \text{on} \quad r = R.$$

The least eigenvalue of this equation corresponds to a symmetric mode, independent of θ, and gives $\alpha^2\lambda_0^* = (\delta_0/R)^2$, where δ_0 is the first zero of $J_0(x)$. It follows that our flow satisfies Arnol'd's criterion, and so is stable, provided we choose α such that

$$\alpha R < \delta_0. \tag{3.8}$$

This Euler flow could be produced by the modified dynamics of Vallis *et al.* provided the initial signature function, which is preserved during the relaxation, is the same as that of the final steady state,

$$V_\omega = \frac{\pi}{\alpha^2}[J_0^{-1}(\omega/\alpha^2 C)]^2.$$

Alternatively, from (3.3), advection of the ψ-lines gives

$$\mathrm{d}^2E = \frac{1}{2}\int_V [(\nabla\phi)^2 - \alpha^2\phi^2]\,\mathrm{d}V,$$

where d^2E is bounded from below by

$$\mathrm{d}^2E_{min} = \lambda_0 \frac{1}{2}\int \phi^2\,\mathrm{d}V.$$

Here λ_0 is the minimum eigenvalue of (3.7),

$$\nabla^2\phi + (\alpha^2+\lambda)\phi = 0, \quad \phi = 0 \quad \text{on} \quad r = R$$

so that d^2E is positive definite if α satisfies $\alpha R < \delta_0$. This is the same condition as (3.8), the requirement for the flow to satisfy Arnol'd's criterion. Thus, stable circular flows can be produced by magnetic relaxation provided the original signature function of the $\boldsymbol{B}$-field matches that of the final flow. (Here the signature function is the volume enclosed by a ψ-line, rather than an ω-line, since it is the ψ-lines which are advected during magnetic relaxation.)

Interestingly, we may also apply Rayleigh's inflexion-point theorem to this simple flow. This states that a sufficient condition for stability of the flow to two-dimensional disturbances is that $\mathrm{d}\omega/\mathrm{d}r$ does not change sign. However,

$$\mathrm{d}\omega/\mathrm{d}r = -u_\theta\,\omega'(\psi) = -\alpha^2 u_\theta$$

so this is satisfied provided α is chosen such that

$$\alpha R < \delta_1,$$

where δ_1 is the first zero of $J_1(x)$. This illustrates the danger of inferring instability from a failure to satisfy Arnol'd's criterion. Flows in the range $\delta_0 < \alpha R < \delta_1$ are stable by Rayleigh's theorem, yet δ^2E is of indefinite sign and so these flows are potentially unstable by Arnol'd's criterion.

Finally, note that we may obtain a closely related flow by setting $\omega'(\psi) = -\alpha^2$. In this case (1.3) gives

$$\psi = C[I_0(\alpha R) - I_0(\alpha r)], \quad u_\theta = \alpha C I_1(\alpha r), \quad \omega = \alpha^2 C I_0(\alpha r),$$

where I_0 and I_1 are modified Bessel functions. Here Rayleigh's inflexion-point theorem states that the flow is always stable, as $u_\theta\,\omega'(\psi)$ does not change sign. This coincides with Arnol'd's criterion which is satisfied since $\delta^2 E$ is positive definite. In addition, such flows may be produced by magnetic relaxation, for any α, since d^2E is always greater than zero.

4. A qualitative explanation of why two-dimensional flows produced by magnetic relaxation are stable

There is no obvious physical reason why flows produced by magnetic relaxation should be stable. On the contrary, there are several good reasons for believing they should, in general, be unstable. In this section, we shall give a simple physical interpretation of the rather formal analysis of §3. First, however, let us consider the counter examples given by Moffatt (1985, 1990).

Suppose we have two thin circular co-axial magnetic flux tubes, one at radius $r = r_1$ and the other at radius $r_2 = r_1 + \delta r$. Let the two tubes have the same volume V, but different magnetic fluxes, Φ_1 and Φ_2. The magnetic energy, M, contained in the tubes is

$$M = \frac{1}{2}\int_{V_1} B_\theta^2 \, dV + \frac{1}{2}\int_{V_2} B_\theta^2 \, dV = \frac{2\pi^2}{V}[\Phi_2^2 r_2^2 + \Phi_1^2 r_1^2].$$

Now suppose that the flux tubes exchange radial position as a result of an axisymmetric perturbation of the 'frozen-field' type. During this exchange the magnetic fluxes and volumes of each tube are conserved, so the change in energy is readily shown to be

$$d^2 M = -\frac{4\pi^2 r_1 (\delta r)^2}{V}\frac{\Phi_2^2 - \Phi_1^2}{\delta r} = -rV\frac{d}{dr}\left(\frac{B_\theta}{r}\right)^2 (\delta r)^2. \tag{4.1}$$

Clearly, fields of the form $(0, B_\theta(r), 0)$ are stable to axisymmetric perturbations only when $|B_\theta/r|$ is a decreasing function of r. This is the familiar interchange instability, and represents the fact that parallel current filaments pinch in on themselves, so that a magnetic flux tube can release energy by contracting.

Now compare this with Rayleigh's circulation criterion. Suppose that we have two thin circular hoops of swirling fluid. If these exchange position, while conserving volume and angular momentum, then an analysis similar to that above (see, for example, Drazin & Reid 1981) gives

$$\delta^2 E = \frac{V}{r^3}\frac{d\Gamma^2}{dr}(\delta r)^2, \quad \Gamma = u_\theta r. \tag{4.2}$$

This is the basis of Rayleigh's circulation criterion for swirling flows subjected to axisymmetric perturbations. For E to be minimal, and the flow to be stable, we require that the angular momentum, Γ, is an increasing function of radius. This is a manifestation of the fact that swirling hoops of fluid can release energy by centrifuging themselves radially outward. We shall see later that a virtual displacement of the vortex lines, in the spirit of Arnol'd or the modified dynamics of Vallis *et al.*, conserves angular momentum, and so expression (4.2) could also be derived from Arnol'd's criterion.

Clearly, the swirling flows $(0, u_\theta(r), 0)$ produced by (axisymmetric) magnetic relaxation will exhibit the characteristic that u_θ/r decreases with radius. Moreover, these Euler flows are, by Rayleigh's circulation theorem, unstable to axisymmetric disturbances. This is one of the simplest counter examples given by Moffatt (1985). It seems probable, therefore, that the stability of the flows discussed in §3 is particular to two dimensions. It is only natural to inquire what special feature of two-dimensional flows allows this to happen.

It is tempting to ascribed the correspondence between magnetic relaxation and Arnol'd's stability criterion to some particular feature of the two dimensional evolution equation for unstable modes. Perhaps, for example, advecting ω is somehow equivalent to advecting ψ. However, it is readily demonstrated that this cannot be true. If ϕ is the perturbation in the streamfunction, then the perturbed equation of motion is

$$\frac{\partial}{\partial t}(\nabla^2\phi)+\boldsymbol{u}\cdot\nabla(\nabla^2\phi+\omega'(\psi)\,\phi)=0.$$

This evolution equation cannot be transformed into an advection equation for ϕ, and so this does not account for our stability results. This becomes evident if we note that the evolution equation above conserves vorticity, while (3.1) does not. That is,

$$\mathrm{d}^1\psi=-\boldsymbol{\eta}\cdot\nabla\psi,\quad \boldsymbol{\eta}=\nabla\times(\epsilon\hat{\boldsymbol{e}}_z)$$

gives

$$\mathrm{d}^1\boldsymbol{u}=(\nabla^2\psi)\,\nabla\epsilon-(\nabla^2\epsilon)\,\nabla\psi+\nabla(\nabla\epsilon\cdot\nabla\psi)-2(\nabla\epsilon)\cdot\nabla(\nabla\psi)$$

from which

$$\int_V \mathrm{d}^1\omega\,\mathrm{d}V=-2\oint_S[(\nabla\epsilon)\cdot\nabla(\nabla\psi)]\cdot\mathrm{d}\boldsymbol{l}.$$

This is not, in general, zero, so global vorticity is not conserved under advection of the ψ-lines. Consequently, we must find a different explanation for the results of §3. In fact, it is the two-dimensional signature function which is the key. This is most readily explained using the well-known analogy between a loaded membrane and two-dimensional flow.

Consider a membrane which is supported along a boundary S and subjected to a load per unit area of $P(x,y)$. Let T be the membrane tension and χ be its equilibrium displacement. We shall assume that the loading P is applied in such a way that it is constant along a displacement contour, $P=P(\chi)$. If we normalize P by introducing $p(\chi)=P(\chi)/T$, then the equation of equilibrium for the membrane is

$$\nabla^2\chi=-p(\chi),\quad \chi=0\quad\text{on}\quad S. \tag{4.3}$$

Now consider vibration of the membrane about its equilibrium position. It we let $\phi\cos(st)$ be the perturbation in displacement, then the equation of motion for the membrane is

$$\nabla^2\phi+p'(\chi)\,\phi+\lambda\phi=0,\quad \lambda=\rho s^2/T, \tag{4.4}$$

where ρ is the density per unit area of the membrane. Now compare these expressions with (1.3) and (3.7). Clearly, there is an exact analogy with two-dimensional Euler flows, with $\psi\leftrightarrow\chi$ and $p\leftrightarrow\omega$. In fact, the analogy carries beyond the equilibrium displacement, stability of the membrane corresponding exactly to Arnol'd's stability criterion. Incidentally, we are now able to attribute physical significance to Arnol'd's and Mestel's functionals $A(\Psi)$ and $M(\Psi)$. The total energy of the membrane, including the stored elastic energy and the potential energy of the load is

$$V(\chi)=T\int\left[\tfrac{1}{2}(\nabla\chi)^2-\int_0^\chi p(\chi)\,\mathrm{d}\chi\right]\mathrm{d}A. \tag{4.5}$$

This is, in effect, Mestel's functional. The fact that Mestel's functional is stationary at equilibrium, and a minimum for stability, corresponds to the theory of minimum potential energy in elastic systems (Sokolnikoff 1946). Arnol'd's functional also has a counterpart in our analogy. The complementary energy of the membrane is defined as (see, for example, Richards 1977)

$$V^*(\chi) = T\iint\left[\tfrac{1}{2}(\nabla\chi)^2 - \int_0^p \chi(p)\,\mathrm{d}p\right]\mathrm{d}A, \tag{4.6}$$

which corresponds exactly to Arnol'd's functional. Again, it is well known that elastic systems exhibit stationary complementary energy at equilibrium. Our analogy is therefore complete.

Returning to the problem in hand, Arnol'd's approach of perturbing the vortex lines corresponds to displacing the contours of constant load (p-lines) and examining the consequent change in strain energy. When $\omega'(\psi) > 0$, Arnol'd's stability criterion translates to a requirement that the strain energy is a maximum under a virtual displacement of the p-lines. Conversely, magnetic relaxation produces equilibria in which the strain energy is a minimum under a virtual displacement of the χ-lines.

We shall now restrict ourselves to the situation where $\omega'(\psi)$ is single signed. Consider first the process of relaxation via 'modified dynamics' in two dimensions. We shall assume that p is initially constant along S, as this is required in the final steady state. When $\omega'(\psi) > 0$, we wish to maximize the strain energy in the membrane by advecting the p-lines (moving the load). This is achieved by pulling the load as close to the centre of the membrane as possible. However, the signature function, V_ω, must be conserved during this process, so the best that can be achieved is that the loading centres itself in the membrane. That is, the p-lines become circular and concentric near the centre of the membrane, as shown in figure 1. (The shape of the p-lines may become modified as we move away from the centre by the need to conform to the boundary.) This is precisely what was observed in the numerical experiments of Vallis *et al.* (1989).

Now consider the process of magnetic relaxation. In our membrane analogy we advect the membrane displacement in such a way as to minimize the stored strain energy, U. Consider the elastic energy stored between two χ-lines, χ and $\chi+\delta\chi$:

$$\delta U = \tfrac{1}{2}T\oint_\chi (\nabla\chi)^2\,\mathrm{d}A.$$

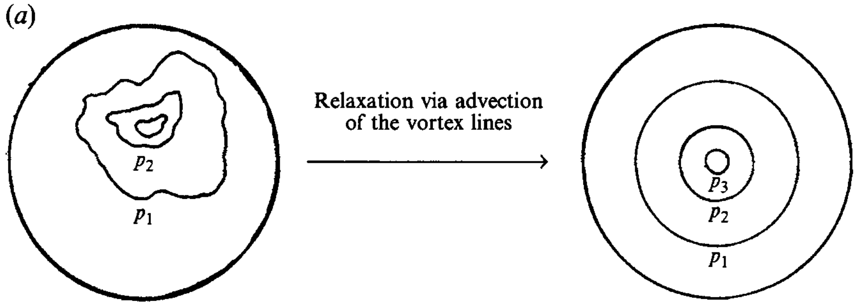
Let l be the distance measured along the χ-line and δn be the separation of the two χ-lines. Then $|\nabla\chi| = \delta\chi/\delta n$ and $\delta A = \mathrm{d}l\,\delta n$. Consequently, we can express the stored energy as

$$\delta U = \tfrac{1}{2}T(\delta\chi)^2\oint_\chi \frac{\mathrm{d}l}{\delta n} = \tfrac{1}{2}T(\delta\chi)^2\oint_\chi \left(\frac{\mathrm{d}l}{\delta A}\right)\mathrm{d}l.$$

Under magnetic relaxation, the area between χ and $\chi+\delta\chi$ is conserved. Consequently, the strain energy is minimized if each χ-line reduces its length, while keeping its enclosed area, $A(\chi)$, constant. This corresponds to magnetic flux tubes loosing energy by contraction (see Moffatt 1985; Linardatos 1993). Under magnetic relaxation, the χ-lines (which correspond to p-lines in the final steady state) will tend to become concentric circles near the centre of the membrane, with their shape modified by the boundary as we move away from the centre. (See figure 1*b*.) Again, this is what was observed in the analytical studies and numerical experiments of Linardatos (1993).

When viewed in terms of the membrane analogy the formal analysis of §3 seems less unreasonable. The key appears to be the constraint imposed by the signature functions. In both cases the p-lines or χ-lines try to centre themselves in the membrane. This is quite different to the axisymmetric situation which led to (4.1) and (4.2). These rely on the existence of axisymmetric perturbations which can exchange the radial position of

(a)



(b)

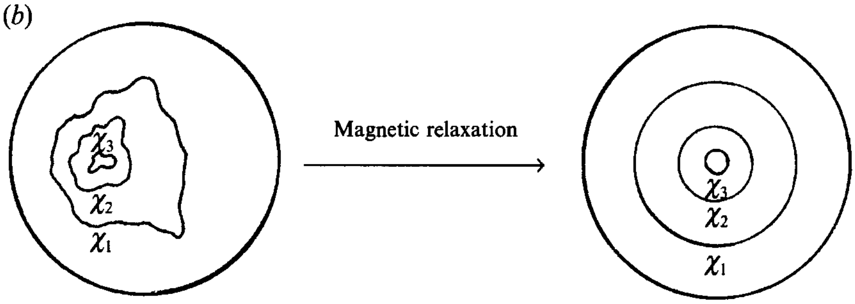


FIGURE 1. Membrane analogy for two-dimensional relaxation via the modified dynamics of Vallis *et al.* and magnetic relaxation. (*a*) The strain energy is maximized by letting the contours of constant load centre themselves in the membrane. (This corresponds to maximizing kinetic energy by advecting the vortex lines.) (*b*) The strain energy is minimized by letting the contours of constant displacement centre themselves in the membrane. (This corresponds to minimizing kinetic energy by advecting the streamlines).

two material hoops. This cannot happen in two dimensions. Here nested streamtubes or flux tubes must remain nested and, in particular, they cannot exchange position, but must remain nested in the same order (see figure 1). This is the key distinction between two-dimensional and three-dimensional perturbations.

5. Magnetic relaxation and Hamilton's principle

We have seen that, under a virtual displacement of the $\boldsymbol{u}$-lines, the kinetic energy of an Euler flow is stationary. However, we have taken a rather circuitous rout to this result, invoking an analogy with magnetohydrodynamics. Since we are dealing with a purely mechanical system, we might anticipate that there is a simpler, strictly mechanical explanation for this. It turns out that there is. The fact that E is stationary under an advection of the ψ-lines follows directly from Hamilton's principle.

The purpose of this section is two-fold. First, it attributes some significance to perturbing the $\boldsymbol{u}$-lines (rather than the ω-lines). Secondly, we demonstrate that, from the point of view of stability, we can completely dispense with magnetic relaxation as an intermediary in our otherwise purely mechanical system. We shall see that, in the framework of Lagrangian dynamics, advecting the $\boldsymbol{u}$-lines can be interpreted as perturbing individual particle trajectories, the extremal nature of E follows from Hamilton's principle, and stability of the flow corresponds to the global action integral being a minimum.

There are two ways in which we could apply Hamilton's principle to our flow. We could consider the motion of individual fluid particles. Alternatively, we could take a global approach and examine the flow field as a whole. Consider first the local approach. There are different ways of stating Hamilton's principle applied to a single particle, but for our purpose the most useful is that given in Lamb (1932). Suppose a particle travels along a trajectory $\boldsymbol{x}_0(t)$, from $\boldsymbol{x}_0(t_1)$ to $\boldsymbol{x}_0(t_0)$, under the influence of a force, $\boldsymbol{F}$. Now consider an adjacent trajectory, $\boldsymbol{x}(t) = \boldsymbol{x}_0(t)+\zeta(t)$. Even though the particle does not follow this new path, we can still calculate the kinetic energy, T, which the particle would have if (somehow) it did. Let $\delta^1 T$ be the first-order difference in kinetic energy between the two paths. Then Hamilton's principle gives us

$$\int_{t_1}^{t_2} (\delta^1 T+\boldsymbol{F}\cdot\boldsymbol{\zeta})\,\mathrm{d}t = 0, \tag{5.1}$$

provided that the perturbed trajectory satisfies

$$[\dot{\boldsymbol{x}}_0(t)\cdot\boldsymbol{\zeta}]_{t_1}^{t_2} = 0. \tag{5.2}$$

(If $\boldsymbol{F}$ is conservative, with potential V, then (5.1) reduces to the more familiar form $\delta\int(T-V)\,\mathrm{d}t = 0$.) Now suppose that $\boldsymbol{x}(t)$ and $\boldsymbol{x}_0(t)$ are both cyclic trajectories, in the sense that they form closed paths and that the particle velocity returns to its original value on completing the circuit. Then provided that the recirculation time, $\tau = t_2-t_1$, is the same for both the true and perturbed paths, condition (5.2) is satisfied and (5.1) follows. Let us now apply this result to a fluid particle moving around a closed streamline in a steady Euler flow. Hamilton's principle gives us

$$\oint_{\psi} \delta^1(\tfrac{1}{2}\rho\boldsymbol{u}^2)\,\mathrm{d}t = \oint_{\psi} (\boldsymbol{\nabla}p\cdot\boldsymbol{\zeta})\,\mathrm{d}t.$$

We merely have to ensure that the recirculation time, $\tau(\psi)$, is the same for all perturbed paths. Now we can create just a set of perturbed trajectories by advecting the ψ-lines by the virtual displacement field $\boldsymbol{\eta}$. Such an approach guarantees that $\tau(\psi)$ is conserved since, from (1.7),

$$\tau(\psi) = -\mathrm{d}V_{\psi}/\mathrm{d}\psi,$$

where V_{ψ} is itself conserved. It is conservation of streamline topology which allows us to make this link. In §2 we saw that ζ and $\boldsymbol{\eta}$ are identical to first order. It follows that

$$\oint_{\psi} \delta^1(\tfrac{1}{2}\rho\boldsymbol{u}^2)\,\mathrm{d}t = \oint_{\psi} \boldsymbol{\nabla}\cdot(p\boldsymbol{\eta})\,\mathrm{d}t. \tag{5.3}$$

Now consider a streamtube bounded by two adjacent streamlines, ψ and $\psi+\delta\psi$. Following the arguments used to derive (1.7), we have

$$\mathrm{d}t = \mathrm{d}l/|\boldsymbol{u}| = \mathrm{d}l\,\delta n/|\delta\psi| = \mathrm{d}A/|\delta\psi|,$$

where l is the distance along the ψ-line and δn is the separation of the two streamlines. Hence, multiplying (5.3) by $\delta\psi$ gives

$$\oint_{\psi} \delta^1(\tfrac{1}{2}\rho\boldsymbol{u}^2)\,\mathrm{d}A = \oint_{\psi} \boldsymbol{\nabla}\cdot(p\boldsymbol{\eta})\,\mathrm{d}A,$$

where the integrals are evaluated over the area of the streamtube. If we now add all such contributions from individual streamtubes, we obtain

$$\int_A \delta^1(\tfrac{1}{2}\rho\boldsymbol{u}^2)\,\mathrm{d}A = \int_S p\boldsymbol{\eta}\cdot\mathrm{d}\boldsymbol{S} = 0. \tag{5.4}$$

Consequently, we conclude that, by virtue of Hamilton's principle, the total kinetic energy is stationary under a volume-preserving displacement of the streamlines.

An alternative and more direct way of reaching the same result is to apply Hamilton's principle to the entire flow field. In this case the pressure force becomes simply a force of constraint whose function is to maintain the non-holonomic constraint, $\nabla \cdot \zeta = 0$ (Lanczos 1970). Now forces of constraint do no net work provided that the constraint which they impose is observed during the virtual displacement. We may, therefore, ignore the pressure forces in this global formulation. As there are no other applied forces, the Lagrangian of the flow is simply the volume integral of the kinetic energy:

$$L = \int (\tfrac{1}{2}\rho \boldsymbol{u}^2)\,\mathrm{d}V = \rho E.$$

the global action integral is then

$$I = \int_{t_1}^{t_2} L\,\mathrm{d}t = \int_{t_1}^{t_2} \rho E\,\mathrm{d}t.$$

The system is monogenic and so we may apply Hamilton's principle directly to the global action integral. In particular, we perturb the system in configuration space by perturbing the particle trajectories with the displacement ζ. Provided our virtual displacement satisfies $\nabla \cdot \zeta = 0$, we know that $\delta^1 I = 0$, demonstrating once again that E is stationary under a volume-preserving advection of the ψ-lines.

Notice that, so far, we have said noting about stability. We have been concerned only with first-order perturbations. Now the arguments of §3 show that the flow is stable provided E is a minimum under advection of the ψ-lines. In terms of I, a two-dimensional Euler flow is two-dimensionally stable provided the global action integral is a minimum at equilibrium. Superficially, this seems plausible. It implies that if (somehow) we perturb the individual particle trajectories by applying a force $\boldsymbol{F}$ to each particle, then a stable flow requires that $\boldsymbol{F}$ perform a net amount of work on the fluid to shift it from its equilibrium configuration. Of course such an argument is rather ill-defined. Interestingly, however, there are many other examples of conservative mechanical systems which, in the absence of external forces (other than those of constraint) and of potential energy, exhibit a minimum kinetic energy at stable equilibrium. A particle moving on the inside surface of a sphere is one simple example. It is interesting to speculate whether or not this is a general principle.

6. Swirling recirculating Euler flows

In the remainder of this paper we shall extend our two-dimensional results to axisymmetric flows, with and without swirl. It is convenient, therefore, to introduce some new notation, and to summarize certain features of axisymmetric Euler flows. We shall adopt a cylindrical polar coordinate system (r, θ, z) and separate the velocity and vorticity into azimuthal (θ) and poloidal (r, z) components:

$$\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_\theta = \nabla \times [(\psi/r)\,\hat{\boldsymbol{e}}_\theta] + (\Gamma/r)\,\hat{\boldsymbol{e}}_\theta.$$

Here ψ is the Stokes streamfunction and Γ is the angular momentum, $u_\theta r$. The streamfunction is related to the vorticity component, ω_θ, by the Stokes operator ∇_*^2, defined by

$$\nabla_*^2 \psi = \frac{\partial^2 \psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = -r\omega_\theta. \tag{6.1}$$

In unsteady flow the Euler equations reduce to transport equations for Γ and ω_θ/r:

$$\mathrm{D}\Gamma/\mathrm{D}t = 0, \tag{6.2}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_\theta}{r}\right) = \nabla\cdot\left[\frac{\Gamma^2}{r^4}\hat{e}_z\right]. \tag{6.3}$$

It is well known that (6.2) and (6.3) admit two classes of steady solution (Batchelor 1967). The first is the swirl-only flow $\Gamma = \Gamma(r)$, $\psi = 0$. The second comes from rewriting (6.3) in the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_\theta}{r} - \frac{\Gamma\Gamma'(\psi)}{r^2}\right) = 0,$$

from which

$$\Gamma = \Gamma(\psi), \tag{6.4}$$

$$\nabla_*^2\psi = -r\omega_\theta = -\Gamma\Gamma'(\psi) + r^2 H'(\psi). \tag{6.5}$$

Equation (6.5) is the counterpart of (1.3) for axisymmetric flows. As before, H is Bernoulli's constant. Given $H(\psi)$ and $\Gamma(\psi)$ we may solve (6.5) for ψ. For confined flows, however, we are restricted in the choice of H and Γ. For example, suppose we specify $\Gamma = \alpha\psi$ and $H'(\psi) = 0$. This is, in fact, a Beltrami flow, with $\boldsymbol{\omega} = \alpha\boldsymbol{u}$ (Davidson 1993). Then (6.5) gives the eigenvalue problem

$$\nabla_*^2\psi + \alpha^2\psi = 0, \quad \psi = 0 \quad \text{on} \quad S, \tag{6.6}$$

which dictates the values of α, and hence Γ, which are compatible with the flow domain. We shall examine the stability of Beltrami flows in §8.

As with two-dimensional flows, we shall find it useful when discussing Hamilton's principle to introduce the recirculation time for a steady axisymmetric flow:

$$\tau(\psi) = \oint_\psi \mathrm{d}l/|\boldsymbol{u}_p|. \tag{6.7}$$

This recirculation time is related to V_ψ, the volume enclosed by the toroidal surface ψ = constant, by the axisymmetric counterpart of (1.7),

$$\tau(\psi) = -\frac{1}{2\pi}\frac{\mathrm{d}V_\psi}{\mathrm{d}\psi}. \tag{6.8}$$

We may also generalize the concept of integral invariants to swirling recirculating flow. Consider the toroidal volume, V_Γ, defined as the volume enclosed by the surface Γ = constant. It is clear from (6.2) that V_Γ is materially advected in an unsteady flow. We may then show (see Davidson 1993) that (6.2) and (6.3) imply the existence of a class of integral invariants of the form

$$I = \int_{V_\Gamma}\left\{\frac{\omega_\theta}{r}h(\Gamma) + f(\Gamma)\right\}\mathrm{d}V, \tag{6.9}$$

where h and f are arbitrary functions of Γ. This invariant will prove useful in generalizing Arnol'd's functional to include flows with swirl. Equation (6.9) also shows that we have two signature functions which characterize axisymmetric flows:

$$V_\Gamma = V_\Gamma(\Gamma) \tag{6.10}$$

and

$$W = \int_{V_\Gamma}\left(\frac{\omega_\theta}{r}\right)\mathrm{d}V = W(\Gamma). \tag{6.11}$$

We shall see that in certain relaxation schemes Γ is a materially advected quantity, and so the signature functions V_Γ and W constrain the relaxation process in the same way that V_ω does in two-dimensional flows.

So far we have assumed that Γ is non-zero. In the absence of swirl ω_θ/r is an advected quantity and we have only one signature function. This is V_ω, the toroidal volume enclosed by the surface ω_θ/r = constant.

Finally, we might note that we can bound the kinetic energy of the flow, just as we did in two dimensions. This time, however, the results depend on whether or not swirl is present. Let us divide the energy E into azimuthal, E_θ, and poloidal, E_p, components. If swirl is absent, conservation of ω_θ/r implies that we may bound E_p from above, but not from below. The axisymmetric version of the Poincaré inequality (1.5) is

$$E_p \leqslant \frac{1}{2\lambda_0}\int_V \left(\frac{\omega_\theta}{r}\right)^2 \mathrm{d}V, \tag{6.12}$$

where λ_0 is the least eigenvalue of

$$\nabla_*^2 \phi + r^2\lambda\phi = 0, \quad \phi = 0 \quad \text{on} \quad S.$$

When swirl is present the situation is rather different. This time conservation of Γ implies that E_θ (and hence E) is bounded from below, but not from above. We can, for example, produce an infinite kinetic energy by pulling a fluid element with finite Γ to the axis. The Schwarz inequality, however, bounds E_θ from below:

$$E_\theta = \int_V \tfrac{1}{2}u_\theta^2\,\mathrm{d}V \geqslant \left\{\int_V \Gamma\,\mathrm{d}V\right\}^2 \Big/ 2\int_V r^2\,\mathrm{d}V. \tag{6.13}$$

The equality holds if, and only if, $u_\theta = \Omega r$, where Ω is a constant. Thus rigid-body rotation represents a minimum-energy state (when Γ is conserved). This lower bound on E ensures that, in principle, the relaxation process of Vallis *et al.* converges to a flow of finite energy (see Davidson 1993).

We shall now look at axisymmetric flow without swirl, essentially as a prelude to examining the more interesting case of swirling recirculating flows.

7. Stability of poloidal flows

We shall see that, when swirl is absent, our stability results for (x, y) flows carry over to (r, z) flows with little modification. In particular, magnetic relaxation provides stable Euler flows and thus an examination of changes in E under advection of the streamlines provides a simple test for stability. As in the preceding sections, we restrict ourselves to two-dimensional (r, z) disturbances. Consider first Arnol'd's criterion. Under advection of the vortex lines,

$$\delta^1\omega_\theta/r = -\boldsymbol{\eta}\cdot\nabla(\omega_\theta/r), \tag{7.1}$$

$$\delta^2\omega_\theta/r = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\delta^1\omega_\theta/r), \tag{7.2}$$

where $\boldsymbol{\eta}(r, z)$ is our virtual displacement field. Now $\boldsymbol{\eta}$ will, in general, contain azimuthal, $\boldsymbol{\eta}_\theta$, and poloidal, $\boldsymbol{\eta}_p$, components. However, $\boldsymbol{\eta}_\theta$ produces no change in $\boldsymbol{\omega}$, and so, without loss of generality we may put $\boldsymbol{\eta}_\theta = 0$. Equation (2.5) then gives the perturbation in kinetic energy as

$$\delta^2 E = \frac{1}{2}\int_V [(\delta^1\boldsymbol{u}_p)^2 - (\boldsymbol{\eta}\cdot\nabla\psi)(\boldsymbol{\eta}\cdot\nabla(\omega_\theta/r))]\,\mathrm{d}V.$$

However, (6.5) tells us that $\omega_\theta/r = -H'(\psi)$, so this integral simplifies to

$$\delta^2 E = \frac{1}{2}\int_V [(\nabla\phi)^2 - (\nabla_*^2\phi)^2/g]\, r^{-2}\, \mathrm{d}V. \tag{7.3}$$

Here ϕ is the first-order perturbation in the streamfunction, $\delta^1\psi$, and

$$g = -r^2 H''(\psi) = r^2 \frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{\omega_\theta}{r}\right). \tag{7.4}$$

There is a direct correspondence between this integral and its x, y analogue. The quantity g now plays the part of $\omega'(\psi)$. Note that we are assuming that g is non-zero everywhere in V, so we are excluding discrete vortex rings. Arnol'd's theorem now tells us that the flow is stable provided $g < 0$. Alternatively, when $g > 0$, we can bound $\delta^2 E$ from above (see Mestel 1989) by

$$\delta^2 E_{max} = (1-\lambda_0)\frac{1}{2}\int (\nabla\phi)^2 r^{-2}\, \mathrm{d}V, \tag{7.5}$$

where λ_0 is the least eigenvalue of

$$\nabla_*^2\phi + g\lambda\phi = 0, \quad \phi = 0 \quad \text{on} \quad S. \tag{7.6}$$

Consequently, as Mestel noted, stability of the flow is guaranteed if either $g < 0$, or else $g > 0$ and the minimum eigenvalue of (7.6) is greater than unity. This is directly analogous to our results for (x, y) flows.

Note that, as with the planar flows, we could also reach the same conclusions by introducing a conserved functional, A, similar to (2.10), but with ω_θ/r replacing ω. In this case (7.3) appears as the second variation of the functional and stability follows if $\delta^2 A$ can be bounded away from zero.

Now consider the Euler flows produced by magnetic relaxation. We start by perturbing the streamlines of the Euler flow, as occurs in the final stage of relaxation. This time, however, we cannot neglect $\boldsymbol{\eta}_\theta$, as it sweeps out an azimuthal component of velocity. Let

$$\boldsymbol{\eta}(r, z) = \boldsymbol{\eta}_\theta + \boldsymbol{\eta}_p.$$

Substituting $\boldsymbol{u}$ for A in (2.3) and (2.4) then gives

$$\mathrm{d}^1\psi = -\boldsymbol{\eta}_p\cdot\nabla\psi, \tag{7.7}$$

$$\mathrm{d}^2\psi = -\tfrac{1}{2}\boldsymbol{\eta}_p\cdot\nabla(\mathrm{d}^1\psi), \tag{7.8}$$

$$\mathrm{d}^1 u_\theta = r\boldsymbol{u}\cdot\nabla(\eta_\theta/r). \tag{7.9}$$

The first two expressions above are to be expected, since ψ is advected in this perturbation. The third term arises from twisting of the streamlines whenever $\boldsymbol{\eta}_\theta$ does not represent rigid-body rotation. It is readily shown that $\mathrm{d}^1 E = 0$, while the second variation in energy is

$$\mathrm{d}^2 E_p + \mathrm{d}^2 E_\theta = \frac{1}{2}\int_V [(\nabla \mathrm{d}^1\psi)^2 + 2\nabla\psi\cdot\nabla(\mathrm{d}^2\psi)]\, r^{-2}\, \mathrm{d}V + \frac{1}{2}\int_V (\mathrm{d}^1 u_\theta)^2\, \mathrm{d}V.$$

Noting that the second term in the integral for $\mathrm{d}^2 E_p$ may be expressed as

$$r^{-2}\nabla\psi\cdot\nabla(\boldsymbol{\eta}_p\cdot\nabla(\mathrm{d}^1\psi)) = \nabla\cdot[-\mathrm{d}^1\psi H'(\psi)\boldsymbol{\eta} + r^{-2}\boldsymbol{\eta}\cdot\nabla(\mathrm{d}^1\psi)\nabla\psi] - H''(\psi)(\mathrm{d}^1\psi)^2$$

the second variation in E simplifies to

$$\mathrm{d}^2 E = \frac{1}{2}\int_V [(\nabla\phi)^2 - g\phi^2]\, r^{-2}\, \mathrm{d}V + \frac{1}{2}\int_V (\mathrm{d}^1 u_\theta)^2\, \mathrm{d}V. \tag{7.10}$$

Here ϕ is the first-order perturbation in the streamfunction ($\phi = \mathrm{d}^1\psi$). Now we know that magnetic relaxation ensures $\mathrm{d}^2E > 0$, for all possible $\boldsymbol{\eta}_p$ and $\boldsymbol{\eta}_\theta$. Since ϕ depends only on $\boldsymbol{\eta}_p$, while d^1u_θ depends only on $\boldsymbol{\eta}_\theta$, it follows that

$$\mathrm{d}^2E_p = \frac{1}{2}\int_V [(\nabla\phi)^2 - g\phi^2]\,r^{-2}\,\mathrm{d}V \tag{7.11a}$$

must be positive for all possible ϕ. This is directly analogous to (3.3) where, once again, g replaces $\omega'(\psi)$. As with the (x,y) flows, Mestel (1989) noted that the conditions for the integral in (7.11a) to be positive definite exactly coincide with the conditions for δ^2E to be of definite sign. The only restriction is that g does not change sign in V. In particular, if $g > 0$,

$$\mathrm{d}^2E_p \geqslant (\lambda_0 - 1)\frac{1}{2}\int g\phi^2 r^{-2}\,\mathrm{d}V, \tag{7.11b}$$

where λ_0 is the minimum eigenvalue of (7.6). Mestel was not concerned with magnetic relaxation, but rather identified the integral in (7.11a) as the second variation of the functional

$$M^*(\Psi) = \int_V [\tfrac{1}{2}r^{-2}(\nabla\Psi)^2 + H(\Psi)]\,\mathrm{d}V,$$

which is stationary when $\Psi = \psi$, the streamfunction of an Euler flow. We conclude, therefore, that magnetic relaxation once again produces Euler flows which are two-dimensionally stable. The only restrictions are that $g \neq 0$ (no isolated vortex rings), g is finite (no vortex sheets), and g is single signed in V.

The other stability results for (x,y) flows also carry over more or less unchanged. The stationary nature of E_p under advection of the ψ-lines follows directly from Hamilton's principle, stability of the flow corresponding to the action integral being a minimum. (This follows from (6.8) and conservation of the signature function V_ψ.) In addition, the eigenvalue problem

$$\nabla^2_*\phi + g\phi + \lambda\phi = 0, \quad \phi = 0 \quad \text{on} \quad S \tag{7.12}$$

gives a simple test for the sign of d^2E_p, for both $g < 0$ and $g > 0$. To ensure $\mathrm{d}^2E_p > 0$, and hence stability of the Euler flow, we require positive eigenvalues of (7.12).

Finally, there is an exact analogy between these results and a simple problem in the theory of elasticity. Consider a shaft of variable diameter, the edges of which are fixed. The shaft is placed in torsion by an azimuthal body force F_θ. Let $v_\theta(r,z)$ be the displacement of the shaft and G be the shear modulus. Then the equilibrium equation for the shaft is (see Den Hartog 1952)

$$\nabla^2_*(rv_\theta) = -rF_\theta/G. \tag{7.13}$$

Let us define χ as the product rv_θ and suppose that the loading is applied such that F_θ/r is constant along χ-contours. For example,

$$F_\theta/r = Gf_\theta(\chi).$$

Then our equilibrium equation becomes

$$\nabla^2_*\chi = -r^2 f_\theta(\chi). \tag{7.14}$$

If this equilibrium is disturbed then the subsequent vibration of the shaft about its equilibrium position is governed by

$$\nabla^2_*\phi + g\phi + \lambda\phi = 0, \quad \lambda = \rho s^2/G, \tag{7.15}$$

where ϕ is the perturbation in χ, s is the natural frequency of oscillation, $g = r^2 f'_\theta(\chi)$, and ρ is the density. If we compare these expressions with (6.5) and (7.12) we see that there is an exact analogy with Euler flows in the (r, z)-plane. The correspondence is between the variables $\psi \leftrightarrow \chi$ and $H'(\psi) \leftrightarrow -f_\theta$. The analogy holds not just for the equilibrium configurations, but also for stability of the two systems, stability of the shaft corresponding exactly to Arnol'd's stability criterion. The total energy of the shaft, including the stored strain energy and the potential energy of the load, is

$$V(\chi) = G \int_V \left[\tfrac{1}{2} r^{-2} (\nabla \chi)^2 - \int_0^\chi f_\theta \, d\chi \right] dV.$$

This is exactly equivalent to Mestel's functional $M^*(\Psi)$, so that the requirement that M^* is a minimum for stable Euler flows corresponds to the theory of minimum potential energy for elastic systems. We may also show that, as for the membrane, Arnol'd's functional for (r, z) flows corresponds to the complementary energy of the shaft, which is necessarily stationary at equilibrium. One advantage of this simple mechanical analogy is that it shows clearly the dependence of the kinetic energy of a flow on the positioning of the vorticity, and so allows one to visualize changes in E which occur during relaxation. In addition, we can avail ourselves of the extensive literature on the stability of elastic systems. We shall now generalize this analysis to include swirl.

8. Arnol'd's theorem applied to swirling recirculating flows

To place this section in context, consider first Rayleigh's stability theorems. Here we are considering axisymmetric rather than two-dimensional perturbations and so Rayleigh's inflexion-point theorem is no longer relevant but must be replaced by his circulation theorem. This tells us that, for flows in which $\boldsymbol{u}_p = \mathbf{0}$, a necessary and sufficient condition for stability is that Rayleigh's discriminant,

$$\Phi(r) = \frac{1}{r^3} \frac{d\Gamma^2}{dr}$$

is positive everywhere in the field of flow. This is consistent with (4.2) which gives $\delta^2 E_\theta = \Phi V (\delta r)^2$, confirming that energy may be transferred from the base flow to the perturbation if $\Phi < 0$. In fact, we may generalize Rayleigh's criterion to Euler flows of the form $(0, u_\theta(r), u_z(r))$. Here Rayleigh's circulation theorem again dictates stability (see Chandrasekhar 1961) with u_z playing no role.

Now flows in which $\boldsymbol{u}_p$ is non-zero inevitably contain regions where Γ decreases with radius. By analogy with (4.2) we might expect that we can then make $\delta^2 E_\theta < 0$ by locally perturbing the flow. In fact, this turns out to be substantially correct. It is tempting, therefore, to conclude that all such flows are unstable. However, there are two problems with this argument. Firstly, a local perturbation of the vortex lines also perturbs E_p, and it is the net change in energy which is important, not just $\delta^2 E_\theta$. Secondly, an unstable oscillation takes a finite time to grow, and could conceivably be swept back and forth between 'stable' and 'unstable' regions of the flow, its amplitude alternatively increasing and decreasing.

We might hope to resolve these questions with a normal mode analysis, which is based on linearized evolution equations for the perturbation. Before embarking on a global approach, let us see what the linearized equations of motion tell us. Suppose we let the perturbations in Γ and ψ be $\delta\Gamma$ and ϕ, and introduce the related variable

$$\gamma = \delta\Gamma / \Gamma'(\psi) - \phi. \tag{8.1}$$

In addition, it is useful to generalize definition (7.4) of g to include swirl:

$$g = \frac{\mathrm{d}}{\mathrm{d}\psi}[\Gamma\Gamma'(\psi)] - r^2 H''(\psi). \tag{8.2}$$

The angular momentum equation (6.2) can be linearized to give

$$\frac{\mathrm{D}\gamma}{\mathrm{D}t} = -\frac{\partial\phi}{\partial t}, \tag{8.3}$$

where $\mathrm{D}/\mathrm{D}t$ is the convective derivative based on the velocity of the steady Euler flow. Equation (8.3) relates perturbations in the azimuthal and poloidal velocities, and we shall make use of this expression later. The transport equation for ω_θ/r gives a second relationship between γ and ϕ:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{\nabla_*^2\phi}{r^2}\right] + \boldsymbol{u}\cdot\nabla\left(\frac{g\phi}{r^2}\right) = -\frac{\partial}{\partial z}\left[\frac{2\Gamma\Gamma'(\psi)\gamma}{r^4}\right]. \tag{8.4}$$

We could, in principle, eliminate ϕ between (8.3) and (8.4) by differentiating (8.4) with respect to time. However, the resulting expression is too complex to allow normal modes to be identified for all but the most trivial of flows. One of the few examples where this does work is the unbounded flow $(0, u_\theta(r), u_z(r))$. In this case normal modes may be found in the form of conventional standing or travelling waves (see Chandrasekhar 1961). In more complex cases, however, perhaps the best we can do is adopt a global approach.

Let us now apply Arnol'd's theorem to this flow. Using (2.3) and (2.4) we may calculate the changes in $\boldsymbol{u}$ arising from a perturbation of the vortex lines. It is natural to express the results in terms of perturbations in Γ and ω_θ/r, and it is readily confirmed that

$$\delta^1\Gamma = -\boldsymbol{\eta}_p\cdot\nabla\Gamma, \tag{8.5}$$

$$\delta^2\Gamma = -\tfrac{1}{2}\boldsymbol{\eta}_p\cdot\nabla(\delta^1\Gamma), \tag{8.6}$$

$$\delta^1\omega_\theta/r = \boldsymbol{\omega}_p\cdot\nabla(\eta_\theta/r) - \boldsymbol{\eta}_p\cdot\nabla(\omega_\theta/r), \tag{8.7}$$

$$\delta^1\boldsymbol{u}_p = (\eta_\theta/r)\nabla\Gamma - (\omega_\theta/r)\nabla\psi^* + \nabla\phi_1. \tag{8.8}$$

Here ψ^* is the streamfunction for $\boldsymbol{\eta}_p$, and ϕ_1 is chosen to ensure that $\delta^1\boldsymbol{u}_p$ is solenoidal. Note that Γ is materially advected in this perturbation, which is to be expected as Γ is the streamfunction for $\boldsymbol{\omega}_p$. The second variation in E_θ is

$$\delta^2 E_\theta = \frac{1}{2}\int_V [(\delta^1\Gamma)^2 + 2\Gamma\delta^2\Gamma]\,r^{-2}\,\mathrm{d}V, \tag{8.9}$$

which, using the identity

$$r^2\nabla\cdot[r^{-2}\Gamma(\boldsymbol{\eta}\cdot\nabla\Gamma)\boldsymbol{\eta}_p] = (\boldsymbol{\eta}\cdot\nabla\Gamma)^2 + \Gamma\boldsymbol{\eta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\Gamma) - (\eta_r/r)\boldsymbol{\eta}\cdot\nabla(\Gamma^2)$$

simplifies to

$$\delta^2 E_\theta = \frac{1}{2}\int_V [(\eta_r/r^3)\boldsymbol{\eta}\cdot\nabla(\Gamma^2)]\,\mathrm{d}V. \tag{8.10}$$

Turning our attention to the poloidal component of velocity we have

$$\delta^2 E_p = \frac{1}{2}\int_V [(\delta^1\boldsymbol{u}_p)^2 + 2\boldsymbol{u}_p\cdot\delta^2\boldsymbol{u}_p]\,\mathrm{d}V$$

and on substituting for $\delta^2 \boldsymbol{u}_p$ we find (see Appendix A)†

$$\delta^2 E_p = \frac{1}{2}\int_V (\delta^1 \boldsymbol{u}_p)^2 \,\mathrm{d}V + \int_V [(\boldsymbol{\eta}\cdot\nabla\Gamma)\,\boldsymbol{u}_p\cdot\nabla(\eta_\theta/r) - \tfrac{1}{2}(\boldsymbol{\eta}\cdot\nabla\psi)\,\boldsymbol{\eta}\cdot\nabla(\omega_\theta/r)]\,\mathrm{d}V. \quad (8.11)$$

Inspection of (8.10) and (8.11) shows that the only possibility of satisfying Arnol'd's criterion is to look for flows where E is a minimum ($\delta^2 E > 0$). This follows from selecting $\boldsymbol{\eta}_p = \boldsymbol{0}$ but $\boldsymbol{\eta}_\theta \neq \boldsymbol{0}$, which gives

$$\delta^2 E = \frac{1}{2}\int_V (\delta^1 \boldsymbol{u}_p)^2 \,\mathrm{d}V > 0.$$

As a simple example, consider the swirl-only flow where $\Gamma = \Gamma(r)$ and $\boldsymbol{u}_p = \boldsymbol{0}$. Then

$$\delta^2 E_\theta = \frac{1}{2}\int_V \Phi \eta_r^2 \,\mathrm{d}V,$$

$$\delta^2 E_p = \frac{1}{2}\int_V (\delta^1 \boldsymbol{u}_p)^2 \,\mathrm{d}V.$$

This is entirely consistent with Rayleigh's circulation theorem. If $\Phi < 0$ at any point, we may ensure $\delta^2 E < 0$ by putting $\boldsymbol{\eta}_\theta = 0$ (so that $\delta^2 E_p = 0$) and then concentrating η_r in the region of negative Φ. In the more general case, where $\psi \neq 0$, our intuition that $\delta^2 E_\theta$ is controlled by the sign of Rayleigh's discriminant is confirmed by (8.10). We may prove this as follows. Suppose that we choose $\boldsymbol{\eta}_p$ to be a local rotation in the (r, z)-plane, applied over a very small area centred on (r_0, z_0). Then

$$\delta^2 E_\theta = \frac{1}{2}\left[\frac{1}{r^3}\frac{\partial \Gamma^2}{\partial r}\right]_0 \int_V \eta_r^2 \,\mathrm{d}V + \frac{1}{2}\left[\frac{1}{r^3}\frac{\partial \Gamma^2}{\partial z}\right]_0 \int_V \eta_r \eta_z \,\mathrm{d}V.$$

However, if $\boldsymbol{\eta}_p$ represents a local rotation, then $\eta_r \eta_z$ will be an odd function in $(r - r_0)$ and $(z - z_0)$. In this case the second integral vanishes, and we are left with a result reminiscent of (4.2):

$$\delta^2 E_\theta = \tfrac{1}{2}\Phi_0 \int_V \eta_r^2 \,\mathrm{d}V. \quad (8.12)$$

This confirms that Rayleigh's discriminant controls the sign of $\delta^2 E_\theta$, even when recirculation is present.

There is, perhaps, a more useful form of $\delta^2 E$. If we define ϵ and γ through the expressions

$$\epsilon = -\boldsymbol{\eta}\cdot\nabla\psi = \delta^1\Gamma/\Gamma'(\psi), \quad \epsilon = 0 \quad \text{on} \quad S, \quad (8.13a)$$

$$\gamma = \epsilon - \delta^1\psi = \epsilon - \phi, \quad \gamma = 0 \quad \text{on} \quad S, \quad (8.13b)$$

then we can rearrange (8.11) in the form (see Appendix A)

$$\delta^2 E = \frac{1}{2}\int_V [(\nabla\gamma)^2]\, r^{-2} \,\mathrm{d}V - \frac{1}{2}\int_V [(\nabla\epsilon)^2 - g\epsilon^2]\, r^{-2} \,\mathrm{d}V. \quad (8.14)$$

Here, we are excluding flows where Γ is constant over a finite region, such as isolated vortex hoops, and also swirl-only flows (where $\boldsymbol{u}_p = \boldsymbol{0}$) since in such cases g is undefined. Note also that ϵ and γ may be varied independently within the class of 'kinematically admissible' functions. This is because ϵ depends solely on $\boldsymbol{\eta}_p$, whereas

† A copy of Appendix A is available from the author or the Editorial Office.

ϕ, and hence γ, depends on both $\boldsymbol{\eta}_p$ and $\boldsymbol{\eta}_\theta$ through (8.7). Of course, within the smaller class of 'dynamically accessible' functions, ϵ and γ are instantaneously linked by the linearized evolution equation (8.3). This gives

$$\frac{\partial \epsilon}{\partial t}+\boldsymbol{u}\cdot\nabla\gamma = 0. \tag{8.15}$$

Here $\boldsymbol{u}$ is the velocity of the steady Euler flow. It is evident from (8.14) that all swirling recirculating flows are potentially unstable to the extent that they fail to meet Arnol'd's criterion. This is true whatever the sign of g. We can always make $\delta^2 E$ negative by choosing $\gamma = 0$ and giving ϵ a short lengthscale so that the second term in (8.14) dominates. If we restrict attention to the smaller class of dynamically accessible functions it may be possible that $\delta^2 E$ is positive definite for certain functions, g. (If this is so, then it seems that large positive values of g promote stability while negative values of g promote instability.) However, as far as Arnol'd's theorem is concerned, all swirling recirculating flows are potentially unstable.

We may illustrate these conclusions with two simple example. Perhaps the simplest, non-trivial example of a swirling flow is the Beltrami flow $\boldsymbol{\omega} = \alpha\boldsymbol{u}$. As discussed in §6, this has the form $\Gamma = \alpha\psi$, $H =$ constant, and $g = \alpha^2$. Here α is an eigenvalue of

$$\nabla_*^2\psi+\alpha^2\psi = 0, \quad \psi = 0 \quad \text{on} \quad S, \tag{8.16}$$

Equation (8.14) then gives the second variation in E as

$$\delta^2 E = \frac{1}{2}\int_V [(\nabla\gamma)^2]\,r^{-2}\,dV-\frac{1}{2}\int_V [(\nabla\epsilon)^2-\alpha^2\epsilon^2]\,r^{-2}\,dV. \tag{8.17}$$

From (7.11a) and (7.11b) it is clear that both integrals above are positive definite. Clearly, such flows are potentially unstable to modes where $\epsilon \gg \gamma$. When the flow is confined to the cylinder $0 \leqslant z \leqslant l$, $r \leqslant R$, equation (8.16) gives

$$\psi = ArJ_1(\delta_n r/R)\sin(m\pi z/l), \quad \alpha^2 = (\delta_n/R)^2+(m\pi/l)^2.$$

Here δ_n is the nth zero of $J_1(x)$. In cases where $l \gg R$ (and $m = 1$) our 'potential instability' translates into a real instability. In particular, the central region of the flow, near $z \sim \frac{1}{2}l$, is of the form $(0, u_\theta(r), u_z(r))$ and so the flow is prone to unstable standing waves of the form discussed earlier. Such waves will initiate in regions where $\Phi < 0$ and have ample time to develop since the recirculation time, $\tau \sim Rl/\Gamma_{max}$, is much greater than the growth time, $T \sim R^2/\Gamma_{max}$.

Another well-known example of a confined swirling flow is Moffatt's (1969) 'vorton' flow. This is a generalization of Hill's spherical vortex. Here the vorticity is confined to a sphere of radius R with $\Gamma = \alpha\psi$ and $H'(\psi) = \lambda$. The rotational flow within the sphere can be matched to an external transverse flow which is uniform at infinity. The parameters α and λ may be varied independently to produce a family of such flows. In all cases $g = \alpha^2$ so that $\delta^2 E$ is again given by (8.17), indicating potential instability to modes with $\epsilon \gg \gamma$.

We conclude this section by showing that an alternative criterion, equivalent to (8.14), may be derived by a more direct route. The approach is essentially a generalization of Arnol'd's functional (2.10). As before we must assume $\Gamma'(\psi)$ is non-zero everywhere in the Euler flow. Consider the functional

$$A(\Psi,\Gamma_t) = \int_V\left[\frac{(\nabla\Psi)^2}{2r^2}+\frac{\Gamma_t^2}{2r^2}-H(\Gamma_t)-\frac{\Omega_\theta}{r}\psi(\Gamma_t)\right]dV, \tag{8.18}$$

where Ψ and Γ_τ are the streamfunction and angular momentum of an unsteady flow, and Ω_θ is the corresponding vorticity, $\nabla_*^2 \Psi = -r\Omega_\theta$. The last two terms in the integral, $H(\Gamma_\tau)$, and $\psi(\Gamma_\tau)$, are functions of Γ_τ which are chosen in a particular way. $\psi(\Gamma_\tau)$ is the inverse function of $\Gamma_0(\psi)$, the angular momentum distribution of some steady Euler flow which is under investigation, and $H(\Gamma_\tau)$ is the Bernoulli function of the same Euler flow. Conservation of energy, in conjunction with (6.9), shows that $A(\Psi, \Gamma_\tau)$ is conserved by any inviscid flow. Now suppose that the unsteady flow (Ψ, Γ_τ) is close to the Euler flow (ψ, Γ_0), so that

$$\Psi = \psi + \phi, \quad \phi \ll \psi,$$
$$\Gamma_t = \Gamma_0 + \delta\Gamma, \quad \delta\Gamma \ll \Gamma_0.$$

It is readily shown that $\delta^1 A = 0$, while the second variation in A is given by an expression identical to (8.14), (see Appendix B);†

$$\delta^2 A = \frac{1}{2}\int_V [g\epsilon^2 - (\nabla\epsilon)^2 + (\nabla\gamma)^2]\, r^{-2}\, \mathrm{d}V. \tag{8.19}$$

Here ϵ and γ are defined as in (8.13*a*) and (8.13*b*): $\epsilon = \delta\Gamma/\Gamma_0'(\psi)$ and $\gamma = \epsilon - \phi$. Now A is conserved by the flow, so $\delta^2 A$ is also conserved in the linear approximation. Following the arguments in §2, stability is ensured if $\delta^2 A$ can be bounded away from zero, for all ϵ and γ. A steady flow is *formally stable* if it possesses an integral invariant which is stationary, and whose second variation is positive or negative definite. Formal stability is a necessary prerequisite to nonlinear stability and, as we have already seen, implies linear stability. Consequently, as with a two-dimensional flow, we can derive the same stability criterion by two different routes. In both the axisymmetric and two-dimensional cases, the use of an integral invariant to establish formal stability is the more direct approach. However, the method of perturbing the vortex lines has the advantage of generality, in that we do not have to look for a new invariant for each class of flow.

The fact that all swirling recirculating flows fail to meet Arnol'd's criterion is, perhaps, just the first indication that all such flows are unstable. Certainly, the existence of regions where $\Phi < 0$ strongly suggests some kind of centrifugal instability. However, we have already seen that one must be wary of implying instability from a failure to satisfy Arnol'd's criterion. Since normal mode analysis is impractical in most cases, we are left with relaxation methods as one means of resolving this issue.

9. Relaxation methods for swirling recirculating flow

There are two established relaxation methods at our disposal: the modified dynamics of Vallis *et al.* and magnetic relaxation. We shall briefly discuss each. Davidson (1993) has already examined the application of modified dynamics to swirling flow. The simplest embodiment of the scheme is to define

$$\hat{\boldsymbol{u}} = \boldsymbol{u} + \lambda\, \partial \boldsymbol{u}_p/\partial t$$

and then 'modified dynamics' replace (6.2) and (6.3) by

$$\hat{\mathrm{D}}\Gamma/\mathrm{D}t = 0,$$
$$\frac{\hat{\mathrm{D}}}{\mathrm{D}t}\left(\frac{\omega_\theta}{r}\right) = \nabla \cdot \left[\frac{\Gamma^2}{r^4}\hat{\boldsymbol{e}}_z\right].$$

† A copy of Appendix B is available from the author or the Editorial Office.

Here $\hat{\mathrm{D}}/\mathrm{D}t$ is the convective derivative based on $\hat{\boldsymbol{u}}$. Note that, as in advection of the vortex lines, angular momentum is a materially conserved quantity. It follows that we may impose a lower bound on E using (6.13). Moreover, this scheme monotonically reduces energy (for $\lambda > 0$) according to

$$\dot{E}(t) = -\lambda \int_V (\partial \boldsymbol{u}_p/\partial t)^2 \,\mathrm{d}V,$$

while preserving the integral invariants given by (6.9). This includes the signature functions (6.10) and (6.11). Consequently, provided we choose $\lambda > 0$, any initial condition is guaranteed to evolve to a stable steady Euler flow. Moreover, conservation of the signature functions might guarantee non-trivial solutions,. However, Davidson (1993) has speculated that such a scheme will minimize E through a radial stratification of the angular momentum, with all the azimuthal vorticity pushed to the boundaries. This simultaneously minimizes both E_θ and E_p. Trivial solutions might then be expected. Attempts to clarify this by numerical experiments were plagued by the growth of singularities in $\boldsymbol{u}$. In fact, no stable solutions were computed.

One might hope that magnetic relaxation, being based on a real physical process, would be better behaved as a relaxation procedure. However, the Euler flows it produces need not be stable. In fact, we have already seen that the swirl-only flow $(0, u_\theta(r), 0)$ produced by axisymmetric relaxation is unstable by Rayleigh's circulation criterion. Physically, this reflects the tendency of flux tubes to lose energy by contraction, whereas swirling vortex hoops release energy by centrifuging themselves radially outward. More generally, we might anticipate that all swirling recirculating flows produced by magnetic relaxation are unstable. We may demonstrate this as follows. If we perturb the $\boldsymbol{u}$-lines of an axisymmetric Euler flow, (2.3) and (2.4) give

$$\mathrm{d}^1\psi = -\boldsymbol{\eta}_p \cdot \nabla\psi, \tag{9.1}$$

$$\mathrm{d}^2\psi = -\tfrac{1}{2}\boldsymbol{\eta}_p \cdot \nabla(\mathrm{d}^1\psi), \tag{9.2}$$

$$\mathrm{d}^1 u_\theta/r = -\boldsymbol{\eta}_p \cdot \nabla(u_\theta/r) + \boldsymbol{u}_p \cdot \nabla(\eta_\theta/r). \tag{9.3}$$

As before, we use d rather than δ to indicate that the streamlines are being perturbed. The first and most important point to note is that Γ is no longer conserved in this perturbation. Rather, it is ψ which is materially advected. It is readily confirmed that $\mathrm{d}^1E = 0$, as would be expected from the magnetostatic equilibrium. This also follows from Hamilton's principle, since (9.1) and (9.3) ensure that the three-dimensional particle trajectories have the same transit time before and after the perturbation is applied. In line with the notation of §7, it is convenient to introduce

$$\phi = -\boldsymbol{\eta} \cdot \nabla\psi, \quad \phi = 0 \quad \text{on} \quad S. \tag{9.4}$$

We may then show (see Appendix C),†

$$\mathrm{d}^2E = \frac{1}{2}\int_V [(\nabla\phi)^2 - g\phi^2]\, r^{-2}\,\mathrm{d}V + \frac{1}{2}\int_V [\mathrm{d}^1\Gamma + \boldsymbol{\eta}_p \cdot \nabla\Gamma]^2 r^{-2}\,\mathrm{d}V. \tag{9.5}$$

Now an Euler flow produced by magnetic relaxation will have $\mathrm{d}^2E > 0$. Moreover, the second integral in (9.5) is a function of both $\boldsymbol{\eta}_\theta$ and $\boldsymbol{\eta}_p$, while the first is only a function

† A copy of Appendix C is available from the author or the Editorial Office.

of $\boldsymbol{\eta}_p$. It follows that magnetic relaxation will give flows in which, for all ϕ,

$$\frac{1}{2}\int_V [(\nabla\phi)^2 - g\phi^2] r^{-2}\,\mathrm{d}V > 0, \quad \phi = -\boldsymbol{\eta}\cdot\nabla\psi. \tag{9.6}$$

One example of a flow which satisfies this is the Beltrami flow discussed in §8. Interestingly, (9.6) has precisely the same form as (7.11*a*), which was derived for magnetic fields and Euler flows without an azimuthal component. The only difference is that we have generalized the definition of g to include u_θ (or B_θ). It follows that the criterion for axisymmetric stability of the magnetostatic equilibrium $(B_r, 0, B_z)$ is precisely the same as for (B_r, B_θ, B_z), but with g extended in definition from (7.4) to (8.2). More importantly, expression (9.6) is precisely the opposite to Arnol'd's stability criterion (8.14). This requires that, for all ϵ,

$$\frac{1}{2}\int_V [(\nabla\epsilon)^2 - g\epsilon^2] r^{-2}\,\mathrm{d}V < 0, \quad \epsilon = -\boldsymbol{\eta}\cdot\nabla\psi. \tag{9.7}$$

Clearly, those modes which represent an increase in the energy of the magnetostatic equilibrium constitute a decrease in the kinetic energy of the corresponding Euler flow. Consequently, as expected on physical grounds, magnetic relaxation will generally produce unstable Euler flows. The key reason is that angular momentum is not conserved during a perturbation of the $\boldsymbol{u}$-lines.

It would appear, therefore, that established relaxation techniques offer little help in finding stable swirling Euler flows. The modified dynamics of Vallis *et al.* are prone to numerical difficulties while magnetic relaxation hunts out the unstable flows. Perhaps this is just one more manifestation of the fact that all such flows are unstable.

10. The Routhian and marginal stability

The analysis of swirling flows simplifies somewhat if we focus on marginally unstable modes. Let us make the tenuous assumption that stable basic flows do indeed exist for certain distributions of g. We know for other distributions of g swirling flows are unstable. A Beltrami flow in an elongated cylinder is one example. In this situation it seems probable that certain basic flows are marginally stable or unstable. Let us make the additional assumption that these marginal modes obey the *principle of exchange of stability*. That is, these unstable modes are not oscillatory, but rather set in as a secondary flow of fixed shape. (The real and imaginary parts of the complex wave frequency are both zero for these marginal modes.) We have no *a priori* justification of this assumption, but it is certainly a hallmark of non-dissipative flows with body forces such as Bénard convection and centrifugal instability in a swirl-only flow (see Chandrasekhar 1961, and Drazin & Reid 1981). We now concern ourselves exclusively with basic flows which are close to, but not at, marginal stability, and we focus on the (almost) marginal modes. In this case the magnitude of the complex wave frequency is much less than the smallest characteristic timescale of the basic flow. Equation (8.15) then gives $\gamma \ll \epsilon$, so that $\delta\Gamma \approx \Gamma'(\psi)\phi$. For these flows and modes we may develop a special variational approach which is essentially a hybrid procedure, retaining aspects of magnetic relaxation (conservation of ψ) and of 'modified dynamics' (conservation of Γ). In many ways this can be traced back to Rayleigh's original work on stability of swirl-only flow.

Rayleigh's first arguments were based on the idea that the centrifugal force in a swirl-only flow may be regarded as an external body force applied to the (incipient) poloidal flow. The details are given in, for example, Rayleigh (1916). In brief, attention is focused on a single fluid particle and it is assumed that this conserves its angular momentum during any perturbation. The centrifugal force, $F_r = \Gamma^2/r^3$, is then conservative with a potential of $\Gamma^2/2r^2$. For swirl-only flows each fluid particle may be regarded as being in a state of static equilibrium and possessing a potential energy of $\frac{1}{2}u_\theta^2$. Stability then corresponds to this potential energy being a minimum. In this way, Rayleigh showed that stability requires that E_θ is a minimum under perturbations which conserve Γ. His celebrated circulation theorem follows. This is essentially Routh's procedure for investigating oscillations about a state of steady motion (Goldstein 1980).

Consider a system with a finite number of degrees of freedom. Routh's procedure in Lagrangian dynamics is frequently used when such systems possess certain types of symmetry. In particular, if one or more generalized coordinates, q, do not appear in the Lagrangian, L, then one can introduce a modified Lagrangian, called the Routhian, defined by

$$R = \sum p_i \dot{q}_i - L, \tag{10.1}$$

Here q_i are the 'ignorable coordinates' which do not appear in L, p_i are the corresponding generalized momenta, which are constants of the motion, and $\dot{q}_i$ represents a Lagrangian time derivative. Provided R is written in such a way that $\dot{q}_i$ does not appear explicitly, then the Routhian satisfies Lagrange's equation for the non-ignorable coordinates. (Here we have used Goldstein's 1980 convention for defining R. Some authors define R to be the negative of the above.) Lamb (1932) has used the Routhian to study the motion of solids in liquids.

Consider now an individual fluid particle in a steady flow. It is subjected to the conservative body force $\boldsymbol{F} = -\nabla(p/\rho)$. For axisymmetric flows θ is an ignorable coordinate, so the corresponding generalized momentum, Γ, is conserved. The Routhian for a fluid particle is then

$$R = \Gamma\dot{\theta} - L = \Gamma\dot{\theta} - (\tfrac{1}{2}\boldsymbol{u}^2 - p/\rho).$$

If $\dot{\theta}$ is now eliminated from R, we obtain,

$$R = \left[\frac{\Gamma^2}{2r^2} + \frac{p}{\rho}\right] - \tfrac{1}{2}\boldsymbol{u}_p^2. \tag{10.2}$$

Provided Γ is treated as a constant, R will satisfy Langrange's equations written in terms of the non-ignorable coordinates, r and z. It follows that we may apply Hamilton's principle to the particle in the (r, z)-plane. We have, in effect, reduced the problem to a strictly two-dimensional one. Note that, as far as the poloidal motion is concerned, the kinetic energy of the swirl now appears as the potential energy of a body force, supplementing p/ρ. This is essentially the approach taken by Rayleigh and we might anticipate that any generalization of Rayleigh's argument would involve R being a minimum. Hamilton's principle now tells us that, for any one fluid particle, the action integral of R around a closed streamline,

$$I = \oint_\psi R \, \mathrm{d}t,$$

will be stationary with respect to perturbations of that particle's trajectory. We merely have to ensure that the perturbed trajectories do not change the recirculation time.

(Note that here we are concerned only with the projection of a particle's trajectory in the (r, z)-plane.) Following the arguments of §5, the pressure term may be dropped from R since, as a force of constraint, it does no net work. We now divide the flow field up into elemental streamtubes (or streamsurfaces), $\delta\psi$, and add the action integral from each. This gives

$$2\pi \sum I(\psi)\,|\delta\psi| = \int_V \left(\frac{\Gamma^2}{2r^2} - \frac{\boldsymbol{u}_p^2}{2}\right) \mathrm{d}V. \tag{10.3}$$

Now suppose that we perturb an Euler flow by advecting the ψ-lines and Γ-lines. This is a hybrid procedure, retaining certain characteristics of both magnetic relaxation and 'modified dynamics'; ψ and Γ are perturbed by a (poloidal) virtual displacement $\boldsymbol{\eta}_p$ in accordance with (2.1) and (2.2). Introducing the symbol Δ to represent this perturbation, we have

$$\Delta^1\psi = -\boldsymbol{\eta}\cdot\nabla\psi, \quad \Delta^2\psi = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\Delta^1\psi), \tag{10.4}$$

$$\Delta^1\Gamma = -\boldsymbol{\eta}\cdot\nabla\Gamma, \quad \Delta^2\Gamma = -\tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla(\Delta^1\Gamma). \tag{10.5}$$

Notice that these perturbations give $\Delta^1\Gamma = \Gamma'(\psi)\,\Delta^1\psi$, so that we are restricting ourselves to marginal modes in flows close to marginal stability. Now under advection of the ψ-lines, the signature function V_ψ is conserved and so, therefore, is the recirculation time given by (6.8). As far as each fluid particle is concerned it finds itself on a new trajectory with the same angular momentum and the same recirculation time. It follows that each action integral in (10.3) will be stationary, and consequently we can introduce a new integral

$$R = \frac{1}{2}\int_V [\Gamma^2 - (\nabla\psi)^2]\, r^{-2}\,\mathrm{d}V = E_\theta - E_p \tag{10.6}$$

which will be stationary under the perturbations defined by (10.4) and (10.5). As far as the poloidal motion is concerned, this represents the difference between the potential energy of the centrifugal force and the kinetic energy of $\boldsymbol{u}_p$. It seems appropriate to refer to this as the Routhian of the flow. For swirl-only flows Rayleigh's criterion requires R to be a minimum for stability. More generally we would expect that a minimum potential energy, and hence a minimum R, would correspond to stability. For brevity, let us now write, $\phi = \Delta^1\psi$. Noting that

$$\Gamma\Delta^1\Gamma - \nabla\psi\cdot\nabla\phi = -r^2\nabla\cdot[(\phi/r^2)\,\nabla\psi + H\boldsymbol{\eta}]$$

we can confirm that R is indeed stationary:

$$\Delta^1 R = \int_V [\Gamma\Delta^1\Gamma - \nabla\psi\cdot\nabla\phi]\, r^{-2}\,\mathrm{d}V = 0.$$

The second variation in R is

$$\Delta^2 R = \frac{1}{2}\int_V [(\Delta^1\Gamma)^2 + 2\Gamma\Delta^2\Gamma - (\nabla\phi)^2 - 2\nabla\psi\cdot\nabla(\Delta^2\psi)]\, r^{-2}\,\mathrm{d}V.$$

We are interested in establishing the conditions under which $\Delta^2 R > 0$. Noting that

$$2\Gamma\Delta^2\Gamma + (\Delta^1\Gamma)^2 = \phi^2[(\Gamma'(\psi))^2 + \Gamma\Gamma''(\psi)] - \Gamma\Gamma'(\psi)\,\boldsymbol{\eta}\cdot\nabla\phi$$

and

$$-2\nabla\psi\cdot\nabla(\Delta^2\psi) = r^2\nabla\cdot[r^{-2}(\boldsymbol{\eta}\cdot\nabla\phi)\,\nabla\psi - \phi H'(\psi)\,\boldsymbol{\eta}] - r^2 H''(\psi)\,\phi^2 + \Gamma\Gamma'(\psi)\,\boldsymbol{\eta}\cdot\nabla\phi$$

it is readily confirmed that $\Delta^2 R$ is given by

$$\Delta^2 R = -\frac{1}{2}\int_V [(\nabla\phi)^2 - g\phi^2]\, r^{-2}\,\mathrm{d}V, \quad \phi = -\boldsymbol{\eta}\cdot\nabla\psi. \tag{10.7}$$

Let us compare this with the second variation in E under advection of the vortex lines. Equation (8.14) tells us that, for flows close to marginal stability ($\gamma \ll \epsilon$),

$$\delta^2 E = -\frac{1}{2}\int_V [(\nabla\epsilon)^2 - g\epsilon^2]\, r^{-2}\,\mathrm{d}V, \quad \epsilon = -\boldsymbol{\eta}\cdot\nabla\psi. \tag{10.8}$$

Perturbing the $\boldsymbol{u}$-lines, on the other hand, results in

$$\mathrm{d}^2 E = \frac{1}{2}\int_V [(\nabla\phi)^2 - g\phi^2]\, r^{-2}\,\mathrm{d}V, \quad \phi = -\boldsymbol{\eta}\cdot\nabla\psi. \tag{10.9}$$

Clearly, for flows and modes close to marginal stability we have $\Delta^2 R = \delta^2 E = -\mathrm{d}^2 E$. Now we know that stable swirling flows exhibit $\delta^2 E > 0$. Thus, as anticipated, a sufficient condition for the stability of these weakly stable or unstable flows is that R is a minimum. This is essentially a generalization of Rayleigh's original analysis.

Of course the problem with (10.7) is that we cannot bound $\Delta^2 R$ from below, so that within the class of kinematically admissible functions, ϕ, the quantity $\Delta^2 R$ is always indefinite in sign or else (as in a Beltrami flow) negative. Perhaps this is just yet another manifestation of the fact that all such flows are unstable.

11. Discussion and conclusions

The content of this paper falls broadly into two parts: two-dimensional flows and swirling flows. In two dimensions Mestel's (1989) work provides the bridge between Arnol'd's stability criterion and magnetic relaxation. The fact that, in two dimensions, magnetic relaxation provides two-dimensionally stable Euler flows of elliptic topology is both surprising and useful. In particular, one would expect that magnetic relaxation, being based on a real physical process, should be better behaved than the modified dynamics of Vallis *et al.* Moreover, perturbing the streamlines provides a simpler mathematical test of stability than that furnished by perturbing the vortex lines. The equivalence of the flows generated by magnetic relaxation and modified dynamics is surprising, since swirling vortex hoops release energy by expanding, while flux tubes release energy by contracting. The key to this equivalence appear to lie in the constraint imposed by the two-dimensional signature function.

The stationary nature of E under advection of the streamlines follows directly from Hamilton's principle. This allows us to dispense with magnetic relaxation as an intermediary in an otherwise purely mechanical system. However, we have not explained why stability of the flow corresponds to the action integral being a minimum. There is, perhaps, scope for further study here.

The analogy between two-dimensional flows and a loaded membrane is more complete than might be expected. Not only do the equilibrium configurations exactly correspond, but so does the stability of the two systems. Mestel's functional is equivalent to the total energy of the membrane, while Arnol'd's functional corresponds to the complementary energy. There are two potential uses of this analogy. Firstly, we can avail ourselves of the extensive literature on the stability of elastic systems. Secondly, we can visualize the effects of advecting ω or ψ on the energy E.

We have seen that, by and large, the stability results for planar flows also apply to poloidal flows. Magnetic relaxation once again produces stable flows and the stationary nature of E under advection of the ψ-lines follows directly from Hamilton's principle. A shaft placed in torsion provides an exact and comprehensive analogy to steady poloidal flows and their stability.

Our attempt to extend the work to include swirl has largely produced negative results. We suspect, but cannot prove, that this reflects the fact that all swirling recirculating Euler flows are unstable. We have shown that, at the very least, they are all potentially unstable to the extent that they fail to meet Arnol'd's criterion. Moreover, both relaxation procedures have failed to unearth any stable flows. Magnetic relaxation produces only unstable flows when swirl is present, and despite initial promise, the modified dynamics of Vallis *et al.* has been dogged by numerical difficulties.

We have seen that, for marginal modes, the concept of the Routhian allows us to extend Rayleigh's original ideas on centrifugal instability. This is essentially a hybrid procedure in which the angular momentum and the streamfunction were both perturbed. We would expect that stable flows correspond to the Routhian being positive definite. In fact, for all swirling, recirculating flows it is indefinite in sign or else negative, again suggesting instability.

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